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NUMERICAL INTEGRATION NEAR A SINGULARITY

By E. L. Kaplan[1]

1. Introduction. The usual formulas for numerical integration, interpolation, etc., assume that the integrand may be approximated by a polynomial. They accordingly give results which are often useless, and at best relatively inaccurate, when employed in the neighborhood of a singularity; in practice this is a point at which the function or one of its derivatives of low order approaches infinity or is discontinuous. By means of polynomial approximation, not of the integrand but of the analytic functions $A(x)$, $B(x)$, the present paper derives and tabulates coefficients for the numerical integration of the functions

(1a) $x^n A(x)$, (1b) $x^{n'}[A(x) \log x + B(x)]$,

beginning at the singularity $x = 0$, for $n = \pm\frac{1}{2}$, and $n' = 0$ or 1, and provides auxiliary tables to facilitate the handling of other values of n. Constant intervals in x are assumed. The tables all extend to a distance of twenty units from the singularity, at which point the integration can easily be handled by the ordinary methods.

Other methods of attacking this problem are the following: (a) changing the variable of integration; (b) expansion of the integrand in series; (c) subtraction (from the integrand) of a function which is integrable by formula, and such that the remainder is integrable numerically. Methods (a) and (b) both require in practice that the complete formal expression of the integrand be known, which the proposed method does not. Method (a) is often unacceptable, and applied to integrand (1b) gives an infinite range of integration. Other objections to (b) are that when considerable accuracy is desired it may be very laborious not only to evaluate the series, but even to calculate a sufficient number of its coefficients; and the process is not readily checked. Method (c) can perhaps not be improved upon when the singularity is a pole and so completely removable by subtracting a polynomial in x^{-1}, but as applied to branch points as in (1a, b) it is a makeshift of low accuracy; here the singularity cannot in general really be removed, and a persistent attempt to do so leads back to method (b). Finally it may be remarked that if the term $A(x) \log x$ can be isolated, it can be integrated by parts (a numerical specification of $A(x)$ is sufficient).

The coefficients c_i tabulated below are used in exactly the same way as the familiar ones (e.g., the $\frac{1}{3}$, $\frac{4}{3}$, $\frac{1}{3}$ of Simpson's rule) and closely resemble the latter, except that they are irrational numbers, and depend not only on the number of ordinates employed and their positions relative to the interval of integration, but also on the nature of the singularity (e.g., value of n) and its position relative to the interval of integration. As the singular point becomes more remote, the values of the new coefficients approach those for polynomials. Series ex-

[1] This paper was prepared in part at Princeton University, under the sponsorship of the Office of Naval Research.

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pansions which demonstrate this approach are readily obtainable, but they are not convenient for purposes of computation.

The advantage of the proposed method over a power-series calculation results primarily from the fact that the rate of convergence of a series depends on the smallness of x itself, while the accuracy of the polynomial approximation depends on its interval of validity (a small multiple of the tabular interval), which may be only a tenth as large.

2. General Theory. Although everything contemplated here can be interpreted in terms of polynomial approximations to analytic auxiliary functions, a more general approach is possible. Let $F_j(x)$ $(j = 0, 1, \cdots, p)$ be a set of functions whose integrals may be calculated in some way. Then if $y(x)$ can be uniformly approximated over the finite interval of integration (a, b) by a linear combination of the F_j whose coefficients are determined by the function-values $y(x_i)(i = 0, 1, \cdots, p)$, $y(x)$ may be integrated numerically by the formula

$$\int_a^b y(x)\,dx \doteq \sum_{i=0}^{p} \bar{c}_i y(x_i), \tag{2a}$$

where the $\bar{c}_i$ are determined from the equations

$$\sum_{i=0}^{p} \bar{c}_i F_j(x_i) = \int_a^b F_j(x)\,dx, \qquad j = 0 \text{ to } p. \tag{2b}$$

By hypothesis there exist constants k_j and ϵ such that

$$|\, y(x) - \sum_j k_j F_j(x) \,| < \epsilon, \qquad \text{whence}$$

$$\left| \int_a^b y(x)\,dx - \sum_j k_j \int_a^b F_j(x)\,dx \right| < \epsilon(b - a). \tag{2c}$$

Substituting from (2b) gives

$$\left| \int_a^b y(x)\,dx - \sum_j \sum_i k_j \bar{c}_i F_j(x_i) \right| < \epsilon\,(b - a) \quad \text{or}$$

$$\left| \int_a^b y(x)\,dx - \sum_i \bar{c}_i y(x_i) \right| < \epsilon\,(b - a),$$

since the k_j were to be determined by the condition that the left member of (2c) be exactly zero for the particular values $x = x_i$. But the actual values of the k_j are not required for the numerical integration by (2a).

Conversely, of course, if the k_j were found, the $\bar{c}_i$ could be dispensed with. The great advantage of the latter is their universal applicability to any function of the class considered. If the required $\bar{c}_i$ have not been tabulated, there is one situation where one might prefer to calculate the k_j, however; namely, when the integrals of a single function over several intervals are required and can be obtained from a single approximating expression (set of k_j), whereas no application of the $\bar{c}_i$ to other functions would be expected. There is of course

no way to use the same c_i for more than one interval of integration. The same considerations apply e.g. to the interpolation of (1b).

3. Example. The tables will be discussed in detail later. As an illustrative application, they have been used to obtain values of

$$\int K'\,dk^2 = 2k^2K' - 2E' + 2 \tag{3}$$

and

$$3\int (E' - 1)\,dk^2 = 2k^2K' - 2(2 - k^2)(E' - 1) - k^2,$$

where K' and E' are the complete elliptic integrals with the modulus $\sqrt{1 - k^2}$. Hayashi's tables[2] were used. The known expansions

$$K' = (1 + \tfrac{1}{4}k^2 + \tfrac{9}{64}k^4 + \tfrac{25}{256}k^6 + \cdots)\ln(4/k) - \tfrac{1}{4}k^2 - \tfrac{21}{128}k^4 - \tfrac{185}{1536}k^6 - \cdots,$$

$$E' = (\tfrac{1}{2}k^2 + \tfrac{3}{16}k^4 + \tfrac{15}{128}k^6 + \cdots)\ln(4/k) + 1 - \tfrac{1}{4}k^2 - \tfrac{13}{64}k^4 - \tfrac{9}{64}k^6 - \cdots$$

show that Table 12 is applicable to either while Table 13 is immediately applicable only to the second (where it should give better results). Argument values $k^2 = .05\ (.05)\ .30$ were used. The first five decimal places always agree and so are not repeated.

Interval	True $\int K'dk^2$	By T.12	Dec. Corr.
0–.05	.16988 62692	47610	5
.05–.10	.13618 26880	26922	8
.10–.15	.12384 44049	44043	9
.15–.20	.11598 89200	89201	10
.10–.30	.45580 77882	77878	9

Interval	True $3\int (E' - 1)dk^2$	By T.13	D.C.	By T.12	D.C.
0–.05	.00498 61864	61656	7	02938	5
.05–.10	.01248 82520	82526	9	82685	7
.10–.15	.01866 60775	60774	10	60755	8
0–.30	.12353 79503	79300	7		

The ordinary integration formulas of course give very inferior results in these cases. The best five and seven-term formulas give $\int_{.10}^{.30} K'\,dk^2 = .45581\ 70643$ and $\int_0^{.30} 3(E' - 1)\,dk^2 = .12344\ 00589$, not quite correct to 5 and 4 decimal places respectively, while the present tables give about 4 additional correct figures.

[2] References, errata and auxiliary tables are given by E. L. Kaplan, J. Math. Phys. **25**: 26–36.

4. The Integrand $x^nA(x)$. In case $F_j(x) = x^{n+j}$ and $x_i = ih$ we write $x = rh$, $a = r_1h$, $b = r_2h$, $c_i = c_ih$. Then the solution of equations (2b) gives

$$c_{\rho+i} = \frac{(\rho+i)^{-n}}{i!\,(p-i)!}\sum_{j=0}^{p}(-)^{i+j}b_{ij}I_{n+j}, \tag{4}$$

where

$$I_{n+j} = \int_{r_1}^{r_2} r^{n+j}\,dr = (r_2^{n+j+1} - r_1^{n+j+1})/(n+j+1) \quad \text{if} \quad n+j+1 \neq 0,$$

and b_{ij} is the elementary symmetric function of degree $(p-j)$ of the p numbers $\rho, \rho+1, \cdots, \rho+i-1, \rho+i+1, \rho+p$. Equation (2a) becomes

$$\int_{r_1h}^{r_2h} y(x)\,dx \doteq h\sum_{i=\rho}^{\rho+p} c_iy(ih) \doteq h^{n+1}\sum_{i=\rho}^{\rho+p}\gamma_iA(ih), \tag{5}$$

where we have introduced the convenient symbol $\gamma_i = i^nc_i$, and $y(x) = x^nA(x)$.

The evaluation of the determinants required in solving (2b) is an elementary exercise. The solution may be verified by substituting it in (2b) and equating the coefficients of I_{n+j}, whence

$$\textstyle\sum_{i=0}^{p}(-)^{i+j}(\rho+i)^kb_{ij}/i!(p-i)! = 1 \qquad \text{if } k = j,$$

and is zero for $k = 0, \cdots, (j-1), (j+1), \cdots, p$. The latter is proved by considering the coefficients of x^j in

$$x^k = \textstyle\sum_{i=0}^{p}(-)^{p-i}(\rho+i)^k\phi_i(x)/i!(p-i)!,$$

where

$$\phi_i(x) = (x-\rho)\cdots(x-\rho-i+1)(x-\rho-i-1)\cdots(x-\rho-p).$$

The identity is established by verifying it for $x = \rho, \rho+1, \cdots, \rho+p$.

If $A(x)$ has a continuous $(p+1)$st derivative throughout the interval considered, the remainder term to be added to its polynomial approximation is known[3] to be

$$[A^{(p+1)}(\xi)\cdot h^{p+1}/(p+1)!](r-\rho)(r-\rho-1)\cdots(r-\rho-p), \tag{6}$$

where ξ is some point in the interval including $x = \rho h$, $x = (\rho+p)h$, and the interval of integration. If the expression $(r-\rho)\cdots(r-\rho-p)$ does not change sign within the interval of integration (e.g. if r_2 and r_1 are consecutive integers) the mean-value theorem may be employed as usual to show that the remainder term for (5) is given rigorously by

$$K_1A^{(p+1)}(\xi)\cdot h^{n+p+2}/(p+1)!, \tag{7}$$

where

$$K_1 = \int_{r_1}^{r_2} r^n(r-\rho)(r-\rho-1)\cdots(r-\rho-p)\,dr = \sum_{j=0}^{p+1} b_jI_{n+j}(-)^{p-j+1}.$$

[3] See for example L. M. Milne-Thomson, The Calculus of Finite Differences.

Here b_j is the elementary symmetric function of degree $(p + 1 - j)$ of the $(p + 1)$ numbers ρ to $\rho + p$.

A result agreeing with (7) must be obtained from a direct determination of the error resulting from the first approximated term (x^{n+p+1}) in the expansion of $y(x)$. This gives

$$K_1 = I_{n+p+1} - \sum_{i=\rho}^{\rho+p} c_i i^{n+p+1}$$

$$= I_{n+p+1} - \sum_{j=0}^{p} I_{n+j} \sum_{i=0}^{p} \frac{(\rho + i)^{p+1}(-)^{i+j}}{i!(p - i)!} b_{ij}. \tag{8}$$

The I's in (7) and (8) must have equal coefficients, so that

$$b_j = \sum_{i=0}^{p} \frac{(\rho + i)^{p+1}}{i!\,(p - i)!} (-)^{p-i} b_{ij}, \tag{9}$$

for $j = 0$ to p.

This result may be proved directly by considering the coefficients of x^j in the identity

$$\phi(x) \equiv x^{p+1} - \sum_{i=0}^{p} \frac{(\rho + i)^{p+1}(-)^{p-i}}{i!\,(p - i)!} \cdot \frac{\phi(x)}{x - \rho - i}, \tag{10}$$

where $\phi(x) = (x - \rho)(x - \rho - 1) \cdots (x - \rho - p)$ and the factor $(x - \rho - i)$ is to be canceled from numerator and denominator. (10) is established by verifying it for $x = \rho, \rho + 1, \cdots, \rho + p$, and comparing coefficients of x^{p+1}. Of course there are also the following simple relations among the b's considered as functions of p and ρ:

$$\begin{aligned} b_j(p, \rho) &= b_{0j}(p + 1, \rho - 1) = b_{p+1,j}(p + 1, \rho) \\ &= (\rho + i)b_{ij}(p, \rho) + b_{i,j-1}(p, \rho) \qquad (i = 0, 1, \cdots, p) \end{aligned} \tag{11}$$

$$\begin{aligned} b_{ij}(p, \rho) &= \rho b_{i-1,j}(p - 1, \rho + 1) + b_{i-1,j-1}(p - 1, \rho + 1) \\ &= (\rho + p)b_{ij}(p - 1, \rho) + b_{i,j-1}(p - 1, \rho). \end{aligned} \tag{12}$$

They are useful for calculating the b's for all but the smallest values of p.

For practical purposes it may be possible to use (7) even when the interval of integration is not confined to a single unit interval. However, caution in this respect is necessary whenever one is working near a zero of $A^{(p+1)}(x)$, or when K_1 is small. It is then possible for example that the the error due to the second approximated term (x^{n+p+2}) of $y(x)$ equals or exceeds that due to the first. Whenever $r_2 - r_1 > 1$, values of

$$K_2 = I_{n+p+2} - \sum_{i=\rho}^{\rho+p} c_i i^{n+p+2} \tag{13}$$

have therefore been tabulated (T.11) as well as K_1. The corresponding component of error is approximately

$$K_2 A^{(p+2)}(\xi) \cdot h^{n+p+2}/(p + 2)!$$

It will be noted that for symmetrical formulas with p even, $K_1 \to 0$ as $\rho \to \infty$, while K_2 has a finite limit, so that at some point K_2 must surpass K_1 as a source of error. Of course, a rigorous, if somewhat pessimistic, upper limit for the error is always obtainable by breaking up the interval of integration into unit intervals, for each of which rigorous limits of error may be obtained by (7). Note that K_1 (and K_2), like the integral itself, is an additive interval function; i.e., if K_1 for the integral from r_1 to r_2 is denoted by $K_1(r_1, r_2)$, then $K_1(r_1, r_2) = K_1(r_1, R) + K_1(R, r_2)$.

5. Restriction to Even (or Odd) Powers. When $A(x)$ is, say, an even function of x, the question arises as to the advantage of approximating it by a linear combination of $(p + 1)$ even powers of x, instead of $(p + 1)$ consecutive powers. The same result is obtained by fitting an unrestricted polynomial of degree $(2p + 1)$ to both the given points $[x_0, A(x_0)], \cdots, [x_p, A(x_p)]$ and their images $[-x_i, A(x_i)]$, so that the remainder term for interpolation is

$$(x^2 - x_0^2) \cdots (x^2 - x_p^2) \cdot A^{(2p+2)}(\xi)/(2p + 2)! \tag{14}$$

The ordinary remainder is $(x - x_0) \cdots (x - x_p) \cdot A^{(p+1)}(\xi)/(p + 1)!$ The reduction in error may be roughly indicated as the ratio $[2\xi/R(p + 1, 2p + 2)]^{p+1}$, where ξ is in the interval including $x, x_0, \cdots, x_p$, and R is the "apparent radius of convergence" of the Maclaurin expansion of $A(x)$ from the $(p + 1)$st to the $(2p + 2)$ terms. This ratio is not notably dependent on the tabular interval. If $A(x)$ is an odd function, the annexed points are $(0, 0)$ and $[-x_i, -A(x_i)]$, and the remainder term is

$$x(x^2 - x_0^2) \cdots (x^2 - x_p^2) A^{(2p+3)}(\xi)/(2p + 3)! \tag{15}$$

Similarly, if the term of lowest degree in $A(x)$ involves $x^m (m > 0)$ it may be approximated by a linear combination of $x^m, x^{m+1}, \cdots, x^{m+p}$. This is equivalent to annexing m confluent argument-values $x = 0$ to the given ones $x_0, \cdots, x_p$, and fitting an unrestricted polynomial of degree $(p + m)$. The remainder term is accordingly

$$x^m (x - x_0) \cdots (x - x_p) A^{(p+m+1)}(\xi)/(p + m + 1)! \tag{16}$$

The reduction of error is suggested by the expression $[\xi/R(p + 1, p + m + 1)]^m$. It is obvious that the integration coefficients for $n = \frac{1}{2}$, say, could be used, with some loss of accuracy, for $n = \frac{3}{2}$ also. The result just obtained shows that this loss of accuracy may be much less (roughly by a factor h/x) than that incurred by decreasing the value of p by one.

When $A(x)$ is approximated by a linear combination of $1, x^2, x^4, \cdots, x^{2p}$ we may add accents to the symbols already used for integration coefficients. An analogous derivation gives

$$\gamma'_{r+i} = (1/D'_i) \sum_{j=0}^{p} (-)^{i+j} b'_{ij} I_{n+2j}, \tag{17}$$

where b'_{ij} is the elementary symmetric function of degree $(p - j)$ of the p numbers $\rho^2, \cdots, (\rho + i - 1)^2, (\rho + i + 1)^2, \cdots (\rho + p)^2$, and

$$D'_i = [(\rho+p)^2 - (\rho+i)^2] \cdots [(\rho+i+1)^2 - (\rho+i)^2]$$
$$\cdot [(\rho+i)^2 - (\rho+i-1)^2] \cdots [(\rho+i)^2 - \rho^2]$$
$$= (p-i)!i!(2\rho+p+i) \cdots (2\rho+2i+1)(2\rho+2i-1) \cdots (2\rho+i) \tag{18}$$
$$= (p-i)!i!(2\rho+p+i)!/(2\rho+i-1)!\cdot(2\rho+2i).$$

When $p = 2$ the formulas are

$$\gamma'_\rho = [I_n(\rho+1)^2(\rho+2)^2 - I_{n+2}(2\rho^2+6\rho+5) + I_{n+4}]/(4\rho+4)(2\rho+1),$$
$$\gamma'_{\rho+1} = [-I_n\rho^2(\rho+2)^2 + I_{n+2}(2\rho^2+4\rho+4) - I_{n+4}]/(2\rho+3)(2\rho+1), \tag{19}$$
$$\gamma'_{\rho+2} = [I_n\rho^2(\rho+1)^2 - I_{n+2}(2\rho^2+2\rho+1) + I_{n+4}]/(2\rho+3)(4\rho+4).$$

6. The Tables. All the tables involve the integers ρ and p as arguments, where ρ and $\rho + p$ are the numbers of tabular intervals h from the singularity to the first and the last of the group of $(p + 1)$ consecutive ordinates under consideration. The range of the tables is $0 \leq \rho,\ \rho + p \leq 20$. The interval of integration (r_1, r_2) and the exponent n are also indicated where required. Tables 1–11 relate to the integrand $x^nA(x)$, while 12–14 concern the logarithmic case. The contents of the tables are as follows (see previous sections for notation):

T.1. $(-)^{i+j}b_{ij}$ for $p = 2, 3, 4$.

T.2. $(-)^{i+j}b'_{ij}$ and D'_i for $p = 2$.

T.3. $(-)^{p-j+1}b_j$ for $p = 4$ (values for $p = 2$ or 3 may be read from T.1 in accordance with equation 11).

T.4. γ'_i for $n = 0$ and 1, γ_i for $n = 2$, all for $p = 2$.

T.5. c_i for $n = \pm\frac{1}{2}$ and $p = 2$.

T.6. c'_i for $n = \pm\frac{1}{2}$ and $p = 2$.

T.7. c_i for $n = \pm\frac{1}{2}$ and $p = 4$.

T.8–10. Like T.5–7, with γ instead of c.

T.11. Selected values of K_1 and K_2, for $p = 2$ and 4.

T.12. c_i for the integrand $A(x) \log x + B(x)$, $p = 4$ or 5.

T.13. c_i for the integrand $x[A(x) \log x + B(x)]$, $p = 5$.

T.14. Inverted matrices for $A(x) \log x + B(x)$.

T.1–4 are exact. T.5–10 are to 13 or 14 decimals, T.11 to 3 significant figures, and T.12–13 to 10 decimals, these being except for T.11 about the number of

reliable figures obtained by using twenty figures in the calculations. This troublesome loss of accuracy is a good omen for the usefulness of the tables, for it reflects the fact that e.g. the function $x^2 \log x$ can be well approximated in a limited range by a linear combination of 1, x, x^2, $\log x$, and $x \log x$. Hence x^3, $x^3 \log x$, etc., should be even better approximated by a linear combination of all six functions, and this is just what is required in practice. This argument also shows why it is probably worthwhile to have the true values of the coefficients to the full number of decimals used in calculation, even though many other sets of values may satisfy the defining equations as closely as the true (rounded) values.

The last couple of groups in T.12–13 are uncertain in the last decimal. For T.14 see the next section. All the coefficients in T.12 have been rounded off so as to give an exact result for a constant function (i.e., they add to $r_2 - r_1$); the others of course do not give exact results for a constant function, although they approach exactness as the singularity recedes (ρ increases).

For the algebraic singularities the value of the multiplier of $x^{\pm 1}$ at $x = 0$ will generally be available for the integration (cf. the starred coefficients in T.5–7). If not known a priori, it can be found by extrapolation, so that formulas for extrapolatory integration need not be given. In the logarithmic case this is not so, since two different analytic functions are being approximated by polynomials. However, extrapolatory integration (third group, T.12) is seen to involve a loss of accuracy between two and three decimals. This loss is reduced almost to one decimal if the dominant term (in $\log x$ alone) is known and integrated separately, by formula (first group). Still, the integral over the one interval adjacent to the singularity is doubtless best obtained from the series expansion, if it is known.

T.5–10 were easily checked by using the two simplest defining equations. This is very adequate since the equations are not themselves used in the calculations, but are solved formally. This is not true of T.12 and 13. Wolfram's table of natural logarithms of integers (checked by the Mathematical Tables Project of New York) was used in the initial calculation, every step being checked. As a final check, the functions $3(3x^2 + 40x + 100) \log_{10} (20/x)$ and $4x(5 + x)(10 + x) \log_{10} (20/x) - x(10 + x)^2 \log_{10} e$ and their integrals $3x(10 + x)^2 \log_{10} (20/x) + (10 + x)^3 \log_{10} e$ and $x^2(10 + x)^2 \log_{10} (20/x)$ were evaluated for $x = 1$ to 20 with the aid of the Logarithmetica Britannica. The linear equations were solved as indicated in the following section, where T.14 is also explained.

7. Interpolation and Integration of $A(x) \log x + B(x)$. In the algebraic cases interpolation presents no difficulty. Only the exponent n being known, multiplying the integrand by x^{-n}, or the integral by x^{-n-1}, gives an interpolable auxiliary function. As usual the logarithmic case is different. Here interpolation is of the same order of difficulty as integration. If it is important to be able to interpolate for values of the integral, the analytic auxiliary functions A and B in (1b) should be determined individually if possible. The corresponding functions for the integral are then $\int_0^x A(x)\,dx$ and

$$\int_0^x \left[B(x) - x^{-1} \int_0^x A(x)\, dx \right] dx$$

respectively, and Table 12 or 13 is not required. However, interpolation is not always important, as when the integration is extended to the singularity primarily to determine a definite integral, either for its own sake or as a check on the indefinite integral.

If interpolation coefficients are required they may be calculated in exactly the same way as the integration coefficients have been, by requiring that they give valid results for each basic function $1, x, x^2, \log x, x \log x, x^2 \log x$. This has not been done because of the great number of coefficients which would have to be tabulated, but it is facilitated by the calculations already carried out. In fact the resulting linear equations will differ from those solved in calculating T.12–13 only in the constant terms, and the same inverse matrix may be used in both cases. Hence these inverse matrices have been reproduced in T.14. They are correct to only about 11 figures owing to the smallness of the determinant, but the ratios of the elements are correct to about the 15 figures given, and these extra figures are required to forestall an additional loss of accuracy when the matrix is multiplied by the vector of constants.

Rather than attempt to invert the sixth-order matrix directly, it was first reduced to third order, taking advantage of the simplicity of the equations corresponding to the functions $1, x, x^2$. The functions actually employed are 1, $x - \rho$, $(x - \rho)^2$, $\log_{10}(100/x)$, $(x/10)\log_{10}(100/x)$, and $(x/10)^2 \log_{10}(100/x)$, where $x = \rho(1)\,\overline{\rho + 5}$ and $\rho = 1, 5, 10, 15$. If the original constant terms corresponding to the foregoing six functions are $\alpha_0, \cdots, \alpha_5$, then form

$$\alpha_0' = \alpha_0 - \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2\,,\quad \alpha_1' = 2\alpha_1 - \alpha_2\,,\quad \alpha_2' = -\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2\,,$$

$$\alpha_{3+i}' = \alpha_{3+i} - \alpha_0'(\rho/10)^i \log_{10}(100/\rho) - \alpha_1'[(\rho+1)/10]^i \log_{10}[100/(\rho+1)] - \alpha_2'[(\rho+2)/10]^i \log_{10}[100/(\rho+2)], \qquad i = 0, 1, 2. \tag{20}$$

The inner product of $(\alpha_3', \alpha_4', \alpha_5')$ with the rows of T.14 gives the last three of the desired coefficients $c_{\rho+3}, c_{\rho+4}, c_{\rho+5}$. The others are given by

$$\begin{aligned} c_\rho &= -c_{\rho+3} - 3c_{\rho+4} - 6c_{\rho+5} + \alpha_0'\,, \\ c_{\rho+1} &= 3c_{\rho+3} + 8c_{\rho+4} + 15c_{\rho+5} + \alpha_1'\,, \\ c_{\rho+2} &= -3c_{\rho+3} - 6c_{\rho+4} - 10c_{\rho+5} + \alpha_2'\,. \end{aligned} \tag{21}$$

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TABLE 1. Values of $b_{ij}(-)^{i+j}$

ρ	$p=3, i=0$				$p=3, i=1$				$p=3, i=2$				$p=3, i=3$			
−1	0	−2	3	−1	2	−1	−2	1	0	2	1	−1	0	−1	0	1
0	6	−11	6	−1	0	6	−5	1	0	−3	4	−1	0	2	−3	1
1	24	−26	9	−1	−12	19	−8	1	8	−14	7	−1	−6	11	−6	1
2	60	−47	12	−1	−40	38	−11	1	30	−31	10	−1	−24	26	−9	1
3	120	−74	15	−1	−90	63	−14	1	72	−54	13	−1	−60	47	−12	1
4	210	−107	18	−1	−168	94	−17	1	140	−83	16	−1	−120	74	−15	1
5	336	−146	21	−1	−280	131	−20	1	240	−118	19	−1	−210	107	−18	1
6	504	−191	24	−1	−432	174	−23	1	378	−159	22	−1	−336	146	−21	1
7	720	−242	27	−1	−630	223	−26	1	560	−206	25	−1	−504	191	−24	1
8	990	−299	30	−1	−880	278	−29	1	792	−259	28	−1	−720	242	−27	1
9	1320	−362	33	−1	−1188	339	−32	1	1080	−318	31	−1	−990	299	−30	1
10	1716	−431	36	−1	−1560	406	−35	1	1430	−383	34	−1	−1320	362	−33	1
11	2184	−506	39	−1	−2002	479	−38	1	1848	−454	37	−1	−1716	431	−36	1
12	2730	−587	42	−1	−2520	558	−41	1	2340	−531	40	−1	−2184	506	−39	1
13	3360	−674	45	−1	−3120	643	−44	1	2912	−614	43	−1	−2730	587	−42	1
14	4080	−767	48	−1	−3808	734	−47	1	3570	−703	46	−1	−3360	674	−45	1
15	4896	−866	51	−1	−4590	831	−50	1	4320	−798	49	−1	−4080	767	−48	1
16	5814	−971	54	−1	−5472	934	−53	1	5168	−899	52	−1	−4896	866	−51	1
17	6840	−1082	57	−1	−6460	1043	−56	1	6120	−1006	55	−1	−5814	971	−54	1
18	7980	−1199	60	−1	−7560	1158	−59	1	7182	−1119	58	−1	−6840	1082	−57	1

ρ	$p=2, i=0$			$p=2, i=1$			$p=2, i=2$			$p=4, i=0$				
−1										0	−6	11	−6	1
0	2	−3	1	0	2	−1	0	−1	1	24	−50	35	−10	1
1	6	−5	1	−3	4	−1	2	−3	1	120	−154	71	−14	1
2	12	−7	1	−8	6	−1	6	−5	1	360	−342	119	−18	1
3	20	−9	1	−15	8	−1	12	−7	1	840	−638	179	−22	1
4	30	−11	1	−24	10	−1	20	−9	1	1680	−1066	251	−26	1
5	42	−13	1	−35	12	−1	30	−11	1	3024	−1650	335	−30	1
6	56	−15	1	−48	14	−1	42	−13	1	5040	−2414	431	−34	1
7	72	−17	1	−63	16	−1	56	−15	1	7920	−3382	539	−38	1
8	90	−19	1	−80	18	−1	72	−17	1	11880	−4578	659	−42	1
9	110	−21	1	−99	20	−1	90	−19	1	17160	−6026	791	−46	1
10	132	−23	1	−120	22	−1	110	−21	1	24024	−7750	935	−50	1
11	156	−25	1	−143	24	−1	132	−23	1	32760	−9774	1091	−54	1
12	182	−27	1	−168	26	−1	156	−25	1	43680	−12132	1259	−58	1
13	210	−29	1	−195	28	−1	182	−27	1	57120	−14818	1439	−62	1
14	240	−31	1	−224	30	−1	210	−29	1	73440	−17886	1631	−66	1
15	272	−33	1	−255	32	−1	240	−31	1	93024	−21350	1835	−70	1
16	306	−35	1	−288	34	−1	272	−33	1	116280	−25234	2051	−74	1
17	342	−37	1	−323	36	−1	306	−35	1	143640	−29562	2279	−78	1
18	380	−39	1	−360	38	−1	342	−37	1					

$p=4$

ρ	$i=1$					$i=2$				
−1	+6	−5	−5	5	−1	0	+6	1	−4	1
0	0	+24	−26	9	−1	0	−12	19	−8	1
1	−60	107	−59	13	−1	40	−78	49	−12	1
2	−240	368	−164	17	−1	180	−216	91	−16	1
3	−630	831	−191	21	−1	504	−450	145	−20	1
4	−1344	920	−230	25	−1	1120	−804	211	−24	1
5	−2520	1459	−311	29	−1	2160	−1302	289	−28	1
6	−4320	2172	−404	33	−1	3780	−1968	379	−32	1
7	−6930	3063	−500	37	−1	6160	−2836	481	−36	1
8	−10560	4216	−626	41	−1	9504	−3900	595	−40	1
9	−15444	5595	−755	45	−1	14040	−5314	721	−44	1
10	−21840	7244	−896	49	−1	20020	−6792	859	−48	1
11	−30030	9187	−1049	53	−1	27720	−8658	1009	−52	1
12	−40320	11448	−1214	57	−1	37440	−10636	1171	−56	1
13	−53040	14051	−1391	61	−1	49504	−13350	1345	−60	1
14	−68544	17020	−1580	65	−1	64260	−16224	1531	−64	1
15	−87210	20379	−1781	69	−1	82080	−19482	1729	−68	1
16	−109440	24152	−1994	73	−1	103360	−23148	1939	−72	1
17	−135660	28363	−2219	77	−1	128520	−27246	2161	−76	1

TABLE 1.—*Continued*

$p = 4$

ρ	$i = 3$					$i = 4$				
−1	0	−3	+1	3	−1	0	+2	−1	−2	1
0	0	+8	−14	7	−1	0	−6	+11	−6	1
1	−30	61	−41	11	−1	24	−50	35	−10	1
2	−144	180	−80	15	−1	120	−154	71	−14	1
3	−420	389	−131	19	−1	360	−342	119	−18	1
4	−960	712	−194	23	−1	840	−638	179	−22	1
5	−1890	1173	−269	27	−1	1680	−1066	251	−26	1
6	−3360	1796	−356	31	−1	3024	−1650	335	−30	1
7	−5544	2605	−455	35	−1	5040	−2414	431	−34	1
8	−8640	3634	−566	39	−1	7920	−3382	539	−38	1
9	−12870	4877	−689	43	−1	11880	−4578	659	−42	1
10	−18480	6388	−824	47	−1	17160	−6026	791	−46	1
11	−25740	8181	−971	51	−1	24024	−7750	935	−50	1
12	−34944	10280	−1130	55	−1	32760	−9774	1091	−54	1
13	−46410	12709	−1301	59	−1	43680	−12122	1259	−58	1
14	−60480	15492	−1484	63	−1	57120	−14818	1439	−62	1
15	−77520	18653	−1679	67	−1	73440	−17886	1631	−66	1
16	−97920	22216	−1886	71	−1	93024	−21350	1835	−70	1
17	−122094	26205	−2105	75	−1	116280	−25234	2051	−74	1

TABLE 2

Values of $b'_{ij}(-)^{i+j}$ *and* D'_i *for* $p = 2$

ρ	b'_{00}	$-b'_{01}$	b'_{02}	D'_0	$-b'_{10}$	b'_{11}	$-b'_{12}$	D'_1	b'_{20}	$-b'_{21}$	b'_{22}	D'_2
0	4	−5	1	4	0	4	−1	3	0	−1	1	12
1	36	−13	1	24	−9	10	−1	15	4	−5	1	40
2	144	−25	1	60	−64	20	−1	35	36	−13	1	84
3	400	−41	1	112	−225	34	−1	63	144	−25	1	144
4	900	−61	1	180	−576	52	−1	99	400	−41	1	220
5	1764	−85	1	264	−1225	74	−1	143	900	−61	1	312
6	3136	−113	1	364	−2304	100	−1	195	1764	−85	1	420
7	5184	−145	1	480	−3969	130	−1	255	3136	−113	1	544
8	8100	−181	1	612	−6400	164	−1	323	5184	−145	1	684
9	12100	−221	1	760	−9801	202	−1	399	8100	−181	1	840
10	17424	−265	1	924	−14400	244	−1	483	12100	−221	1	1012
11	24336	−313	1	1104	−20449	290	−1	575	17424	−265	1	1200
12	33124	−365	1	1300	−28224	340	−1	675	24336	−313	1	1404
13	44100	−421	1	1512	−38025	394	−1	783	33124	−365	1	1624
14	57600	−481	1	1740	−50176	452	−1	899	44100	−421	1	1860
15	73984	−545	1	1984	−65025	514	−1	1023	57600	−481	1	2112
16	93636	−613	1	2244	−82944	580	−1	1155	73984	−545	1	2380
17	116964	−685	1	2520	−104329	650	−1	1295	93636	−613	1	2664
18	144400	−761	1	2812	−129600	724	−1	1443	116964	−685	1	2964

TABLE 3

Values of $b_i(-)^{p-i+1}$ *for* $p = 4$ *(for* $p < 4$, *see eqn.* 11)

p	$-b_0$	b_1	$-b_2$	b_3	$-b_4$	b_5
−1	0	−6	+5	5	−5	1
0	0	+24	−50	35	−10	1
1	−120	274	−225	85	−15	1
2	−720	1044	−580	155	−20	1
3	−2520	2754	−1175	245	−25	1
4	−6720	5944	−2070	355	−30	1
5	−15120	11274	−3325	485	−35	1
6	−30240	19524	−5000	635	−40	1
7	−55440	31594	−7155	805	−45	1
8	−95040	48504	−9850	995	−50	1
9	−154440	71394	−13145	1205	−55	1
10	−240240	101524	−17100	1435	−60	1
11	−360360	140274	−21775	1685	−65	1
12	−524160	189144	−27230	1955	−70	1
13	−742560	249754	−33525	2245	−75	1
14	−1028160	323844	−40720	2555	−80	1
15	−1395360	413274	−48875	2885	−85	1
16	−1860480	520024	−58050	3235	−90	1
17	−2441880	646194	−68305	3605	−95	1

TABLE 4

p	r_1	r_2	$n = 0, p = 2$				$n = 1, p = 2$				$n = 2, p = 2$			
			$\gamma'_p\Delta_1$	$\gamma'_{p+1}\Delta_1$	$\gamma'_{p+2}\Delta_1$	Δ_1	$\gamma'_p\Delta_2$	$\gamma'_{p+1}\Delta_2$	$\gamma'_{p+2}\Delta_2$	Δ_2	$\gamma_p\Delta_2$	$\gamma_{p+1}\Delta_2$	$\gamma_{p+2}\Delta_2$	Δ_2
0	0	2	12	64	14	45	−3	16	5	9	−2	24	18	15
2	2	4	518	2112	520	1575	21	144	45	35	18	184	78	15
4	4	6	2464	9920	2466	7425	385	2000	585	297	78	504	178	15
6	6	8	6810	27328	6812	20475	1155	5488	1547	585	178	984	318	15
8	8	10	14516	58176	14518	43605	855	3888	1071	323	318	1624	498	15
10	10	12	26542	106304	26544	79695	4807	21296	5775	1449	498	2424	718	15
12	12	14	43848	175552	43850	131625	8073	35152	9425	2025	718	3384	978	15
14	14	16	67394	269760	67396	202275	4185	18000	4785	899	978	4504	1278	15
16	16	18	98140	392768	98142	294525	18445	78608	20757	3465	1278	5784	1618	15
18	18	20	137046	548416	137048	411255	25035	100744	26823	4329	1618	7224	1998	15
0	0	1	57	34	−1	90	33	40	−1	144	7	36	−3	120
1	1	3	145	608	147	450	10	128	42	45	3	84	43	15
3	3	5	1251	5056	1253	3780	180	1024	306	180	43	324	123	15
5	5	7	4277	17184	4279	12870	234	1152	330	143	123	724	243	15
7	7	9	10183	40832	10185	30600	1768	8192	2280	765	243	1284	403	15
9	9	11	19929	79840	19931	59850	3570	16000	4370	1197	403	2004	603	15
11	11	13	34475	138048	34477	103500	2100	9216	2484	575	603	2884	843	15
13	13	15	54781	219296	54783	164430	10150	43904	11718	2349	843	3924	1123	15
15	15	17	81807	327424	81809	245520	15312	65536	17360	3069	1123	5124	1443	15
17	17	19	116513	466272	116515	349650	7326	31104	8190	1295	1443	6484	1803	15
18	19	20	−74139	560246	337403	823510	−19773	148200	90909	11348	−3303	30036	19407	120
4	5	6	−1661	10070	5841	14850	−1331	8360	4851	2160	−303	2316	1627	120
8	9	10	−8569	60806	35173	87210	−6859	47880	28747	7344	−903	7036	4707	120
14	15	16	−37231	276510	165271	404550	−29791	219480	133951	20880	−2403	18916	12327	120

TABLE 5a

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x)\,dx$

p	r_1	r_2	c_p	c_{p+1}	c_{p+2}
0	0	2	1.1313708498* 9848	1.5084944665 3130	.2666666666 6667
2	2	4	.3267724651 7765	1.3448511938 3688	.3281193008 7987
4	4	6	.3311548472 7528	1.3373842708 3113	.3314297616 5681
6	6	8	.3322530319 3466	1.3353872882 0696	.3323517242 7485
8	8	10	.3326891332 9407	1.3345726913 6486	.3327352853 6654
10	10	12	.3329057937 7524	1.3341619221 5349	.3329309938 0066
12	12	14	.3330289768 5659	1.3339261464 3615	.3330442166 9357
14	14	16	.3331056611 2036	1.3337783963 2226	.3331155706 9550
16	16	18	.3331566217 9087	1.3336797288 4650	.3331634241 6490
18	18	20	.3331922020 5520	1.3336105817 4101	.3331970719 6765
0	0	1	1.2000000000* 0000	.9333333333 3333	−.1885618083 1641
1	1	3	.3164089278 5952	1.3605470444 0647	.3215390309 1734
3	3	5	.3298334620 7417	1.3397090264 3009	.3303799124 5623
5	5	7	.3318460105 3843	1.3361355355 6235	.3320036150 0640
7	7	9	.3325129814 7497	1.3349035178 4303	.3325788576 3183
9	9	11	.3328140249 8661	1.3343364727 3609	.3328476102 0965
11	11	13	.3329752046 6056	1.3340292896 3392	.3329945957 7953
13	13	15	.3330714812 5020	1.3338443527 8573	.3330836756 2861
15	15	17	.3331335574 6335	1.3337244360 2606	.3331417193 9611
17	17	19	.3331759093 0893	1.3336422734 1757	.3331816381 6231
18	19	20	−.0800680256 4670	.6602225426 4029	.4198476940 5538
4	5	6	−.0711110693 8438	.6434804521 7612	.4277442494 8339
8	9	10	−.0764878125 7691	.6533615595 8896	.4231464585 4220
14	15	16	−.0792046768 4000	.6585481352 4460	.4206610020 6990

* These are values of γ_p.

TABLE 5b

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{1/2} A(x)\,dx$

ρ	r_1	r_2	c_ρ		$c_{\rho+1}$		$c_{\rho+2}$	
0	0	2	.1077496047*	5224	1.2029952570	2683	.3428571428	5714
2	2	4	.3361286282	2335	1.3295726435	6205	.3347351072	1538
4	4	6	.3341687865	0766	1.3319927724	2818	.3338906327	3518
6	6	8	.3337306934	6291	1.3326511907	1995	.3336314166	9651
8	8	10	.3335650071	3625	1.3329211278	0949	.3335186907	4686
10	10	12	.3334849337	3336	1.3330575455	8548	.3334596738	3236
12	12	14	.3334402143	1458	1.3331359380	9710	.3334249485	9795
14	14	16	.3334127227	5022	1.3331850968	8724	.3334028005	4939
16	16	18	.3333946225	3402	1.3332179395	7451	.3333878134	1745
18	18	20	.3333820756	6144	1.3332409629	4614	.3333772018	8664
0	0	1	.2095238095*	2381	.5142857142	8571	−.0404061017	8209
1	1	3	.3417578350	5904	1.3246982760	5358	.3361742293	1265
3	3	5	.3347275397	9308	1.3312322351	6474	.3341708487	5630
5	5	7	.3338898926	0763	1.3324039345	1505	.3337310106	3335
7	7	9	.3336312627	8141	1.3328114050	8767	.3335650890	4271
9	9	11	.3335186439	4365	1.3329995522	6403	.3334849620	3710
11	11	13	.3334596559	0673	1.3331016361	2300	.3334402261	1176
13	13	15	.3334249405	7820	1.3331631489	2381	.3334127283	5541
15	15	17	.3334027965	3659	1.3332030568	0267	.3333946254	6776
17	17	19	.3333878112	3257	1.3332304101	0985	.3333820773	1558
18	19	20	−.0867346575	4314	.6732001886	4502	.4135231717	2099
4	5	6	−.0976967831	3384	.6910384227	1544	.4060371051	0543
8	9	10	−.0909041085	1361	.6803566166	6303	.4103415786	5416
14	15	16	−.0876817654	2092	.6749274264	8954	.4127314117	9311

* These are values of γ_0.

TABLE 6a

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x^2)\,dx$

ρ	r_1	r_2	c'_ρ		$c'_{\rho+1}$		$c'_{\rho+2}$	
0	0	2	1.2570787221*	0942	1.3408839702	5005	.3259259259	2593
2	2	4	.3305606214	4011	1.3368977224	8634	.3319459165	6756
4	4	6	.3325001676	9727	1.3346495181	8104	.3327778588	0090
6	6	8	.3329365275	1526	1.3340092036	1211	.3330357259	6109
8	8	10	.3331018559	0329	1.3337432543	3279	.3331481508	6051
10	10	12	.3331818193	4778	1.3336080997	7248	.3332070715	4702
12	12	14	.3332264961	4258	1.3335302057	4248	.3332417585	5706
14	14	16	.3332539684	2675	1.3334812750	7096	.3332638890	2476
16	16	18	.3332720589	0382	1.3334485485	6174	.3332788671	6735
18	18	20	.3332846004	3056	1.3334255893	4886	.3332894737	1789
0	0	1	1.5555555555*	5556	.4592592592	5926	−.0209513120	3516
1	1	3	.3250988610	3139	1.3408839702	5005	.3305698743	7817
3	3	5	.3319451666	1439	1.3353741173	2877	.3325002893	3085
5	5	7	.3327778298	0965	1.3342510613	4314	.3329365363	7058
7	7	9	.3330357227	4233	1.3338515951	6775	.3331018572	3439
9	9	11	.3331481502	5237	1.3336656175	8459	.3331818196	4855
11	11	13	.3332070713	8834	1.3335643099	8824	.3332264962	3093
13	13	15	.3332417585	0560	1.3335031292	2373	.3332539684	5792
15	15	17	.3332638890	0524	1.3334633822	8030	.3332720589	1642
17	17	19	.3332788671	5902	1.3334361145	4547	.3332846004	3621
18	19	20	−.0865998903	3507	.6733001023	8145	.4133155907	9060
4	5	6	−.0953677469	9098	.6927891613	0037	.4034375064	4146
8	9	10	−.0901606235	7747	.6808394237	1128	.4094689140	0819
14	15	16	−.0874619905	7858	.6750009708	6175	.4124030746	0998

* These are values of γ'_0.

TABLE 6b

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{1/2} A(x^2)\,dx$

p	r_1	r_2	c'_p		c'_{p+1}		c'_{p+2}	
0	0	2	−.0979541861*	3840	1.5672669782	1434	.2943722943	7229
2	2	4	.3197651183	7563	1.3524767861	4943	.3264702866	6014
4	4	6	.3291961521	5727	1.3400794258	3528	.3305687008	7084
6	6	8	.3313559250	8552	1.3367552595	3932	.3318490215	8597
8	8	10	.3321782164	0425	1.3353984561	3962	.3324088740	7520
10	10	12	.3325767397	6615	1.3347141053	5706	.3327027024	7214
12	12	14	.3327996338	9507	1.3343212487	4529	.3328758169	0545
14	14	16	.3329367776	4494	1.3340750452	5255	.3329863176	2966
16	16	18	.3330271215	8342	1.3339106231	1136	.3330611292	3952
18	18	20	.3330897703	4163	1.3337953909	4816	.3331141174	9001
0	0	1	.3549783549*	7835	.3203463203	4632	−.0061221366	3365
1	1	3	.2943886301	2138	1.3782904137	0292	.3197650553	7941
3	3	5	.3264702892	0518	1.3439451040	9726	.3291961519	0993
5	5	7	.3305687009	1011	1.3380011394	7054	.3313559250	7695
7	7	9	.3318490215	8830	1.3359495396	4071	.3321782164	0351
9	9	11	.3324088740	7546	1.3350049239	5665	.3325767397	6604
11	11	13	.3327027024	7219	1.3344931133	7906	.3327996338	9504
13	13	15	.3328758169	0546	1.3341849535	2502	.3329367776	4494
15	15	17	.3329863176	2966	1.3339851251	0355	.3330271215	8342
17	17	19	.3330611292	3952	1.3338482048	0813	.3330897703	4163
18	19	20	−.0938227176	5577	.6866444154	1800	.4071436846	1730
4	5	6	−.1312394031	2692	.7455864797	6850	.3836292041	1840
8	9	10	−.1070954332	5529	.7094590412	5643	.3973040028	0813
14	15	16	−.0968429988	2718	.6920576051	6694	.4047147367	7095

* These are values of γ'_0.

TABLE 7a

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x)\,dx$

ρ	r_1	r_2	c_ρ	$c_{\rho+1}$	$c_{\rho+2}$	$c_{\rho+3}$	$c_{\rho+4}$
−1	0	2	−.0329234903* 4096	1.2271482763* 4491	1.4187031292 3777	.3005291005 2910	+.0051841052 7573
1	2	4	−.0087403039 9604	.3699776489 4906	1.2769371156 8064	.3715785075 0623	−.0096816520 8440
3	4	6	−.0104029724 3597	.3751855331 2236	1.2702819388 4730	.3755126340 1840	−.0105747137 6310
5	6	8	−.0107705745 5728	.3764920020 1669	1.2685001353 4267	.3766094605 9711	−.0108307253 6336
7	8	10	−.0109110260 2754	.3770109795 8150	1.2677730837 1921	.3770659134 2454	−.0109388866 5022
9	10	12	−.0109794727 9985	.3772688493 2203	1.2674064099 3152	.3772988460 3278	−.0109946135 5933
11	12	14	−.0110179384 6295	.3774154726 5094	1.2671959291 9022	.3774336138 7839	−.0110270703 6848
13	14	16	−.0110416979 7817	.3775067529 0182	1.2670640246 7351	.3775185493 1444	−.0110476260 0814
15	16	18	−.0110573990 5046	.3775674150 2718	1.2669759364 6594	.3775755127 1725	−.0110614637 9459
17	18	20	−.0110683151 0497	.3776097695 4769	1.2669142022 5952	.3776155668 4811	−.0110712228 5158
−1	0	1	−.0455026455* 0265	1.3121693121* 6931	.8698412698 4127	−.2274713878 1028	+.0421557339 4083
0	1	3	−.0124341561* 6106	.3576959167 7061	1.2908884990 8150	.3637797966 1228	−.0079690408 5579
2	3	5	−.0099357252 5320	.3736142985 4492	1.2723551845 3023	.3742642749 6739	−.0102879098 6319
4	5	7	−.0106365673 0119	.3760077154 8851	1.2691678511 9614	.3761952693 8240	−.0107334959 9611
6	7	9	−.0108547042 1637	.3768013406 8278	1.2680683720 4890	.3768797482 1180	−.0108946253 2762
8	9	11	−.0109505985 4225	.3771596230 3918	1.2675622268 9204	.3771996000 8974	−.0109708177 3616
10	11	13	−.0110011905 3295	.3773514673 1006	1.2672880081 2779	.3773745498 3058	−.0110128235 5963
12	13	15	−.0110311266 9285	.3774660669 0680	1.2671229081 1148	.3774805830 3388	−.0110384269 5661
14	15	17	−.0110503021 6076	.3775399597 4602	1.2670158503 1730	.3775496758 2556	−.0110551817 6066
16	17	19	−.0110633214 5851	.3775903746 8523	1.2669424965 5526	.3775971944 6768	−.0110667432 8201
16	19	20	−.0238725586 4107	.1373018907 1962	−.3519914326 7686	.8875682503 2549	.3509938798 6693
2	5	6	−.0158468283 3557	.1083318249 0696	−.3119187216 9881	.8625995209 8463	.3568824489 1144
6	9	10	−.0209180520 3699	.1260874179 3469	−.3359426557 6423	.8773164242 2951	.3534584365 2393
12	15	16	−.0231815342 4448	.1346331490 8626	−.3481174294 0275	.8850637923 6846	.3516021238 5445
0	0	2	1.0625308246* 4010	1.7479380326 4738	−.1650793650 7937	.2903098954 4097	−.0658469806 8192
0	0	1	1.0846560846* 5608	1.3248677248 6772	−.8709759717 4724	.4362202033 8771	−.0910052910 0529

* These are values of γ_{-1} and γ_0.

TABLE 7b

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x^{1/2} A(x)\,dx$

ρ	r_1	r_2	c_ρ	$c_{\rho+1}$	$c_{\rho+2}$	$c_{\rho+3}$	$c_{\rho+4}$
−1	0	2	+.0016325697* 6897	.1251636822* 8795	1.2309576058 0585	.3890331890 3319	−.0128817161 3970
1	2	4	−.0126738951 8956	.3811035249 7838	1.2633107560 7400	.3794456898 9320	−.0114408075 4526
3	4	6	−.0114383436 1118	.3787721640 8267	1.2654699684 6480	.3784411059 3455	−.0112525233 0349
5	6	8	−.0112522351 4552	.3782507778 1165	1.2660576703 6972	.3781326051 1511	−.0111897339 6916
7	8	10	−.0111896682 3470	.3780535632 5421	1.2662986477 4935	.3779984286 1030	−.0111611662 4510
9	10	12	−.0111611450 8828	.3779582471 8502	1.2664204373 2432	.3779281773 0054	−.0111457740 4039
11	12	14	−.0111457656 1969	.3779050133 9082	1.2664904258 9621	.3778868405 3503	−.0111365350 0569
13	14	16	−.0111365311 3227	.3778722869 5729	1.2665343155 0237	.3778604751 1413	−.0111305551 9569
15	16	18	−.0111305532 1641	.3778507398 9362	1.2665636382 2453	.3778426339 6303	−.0111264629 7675
17	18	20	−.0111264618 8121	.3778358035 8440	1.2665841941 9559	.3778300015 6353	−.0111235395 6518
−1	0	1	−.0170274170* 2742	.2501202501* 2025	.4946608946 6089	−.0506096628 3817	+.0060539855 0665
0	1	3	−.0075240006* 5872	.3877919194 0550	1.2589668677 5690	.3811557318 3507	−.0117310411 8515
2	3	5	−.0117185601 0927	.3794370128 0226	1.2647912568 3148	.3787745300 3707	−.0113172300 4727
4	5	7	−.0113164832 0765	.3784402565 8581	1.2658369571 1851	.3782511418 4059	−.0112142691 4420
6	7	9	−.0112141389 9272	.3781324284 4475	1.2662006941 6502	.3780536572 7559	−.0111730423 0818
8	9	11	−.0111730062 1071	.3779983748 8212	1.2663686620 9512	.3779582796 7754	−.0111524128 3703
10	11	13	−.0111523997 7230	.3779281567 2155	1.2664598010 1482	.3779050269 3446	−.0111406186 9163
12	13	15	−.0111406130 6651	.3778868313 2792	1.2665147199 9921	.3778722933 9243	−.0111332449 3756
14	15	17	−.0111332421 9987	.3778604705 0711	1.2665503505 0229	.3778507432 6179	−.0111283279 9973
16	17	19	−.0111283265 4006	.3778426314 5458	1.2665747722 8346	.3778358054 8350	−.0111248855 9451
16	19	20	−.0291713750 0168	.1578643185 2722	−.3819660645 2369	.9070183598 2975	.3462544744 3388
2	5	6	−.0439613095 4349	.2001533360 3024	−.4311983635 6876	.9337241576 6543	.3406505456 1792
6	9	10	−.0332949504 2439	.1719225431 3427	−.4002549063 0551	.9177392050 9272	.3438713322 5750
12	15	16	−.0300415017 0733	.1609965232 8597	−.3862240039 6806	.9096067580 8102	.3456612177 6250
0	0	2	+.1333265311* 3282	1.2146319081 1611	+.4005772005 7720	−.0175945391 1763	.0008162848 8449
0	0	1	+.1649831649* 8316	.6649350649 3506	−.1710116832 9995	.0552079045 2857	−.0085137085 1371

* These are values of γ_{-1} and γ_0.

TABLE 8a

Coefficients (for $A(x)$) for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x)\,dx$

ρ	r_1	r_2	γ_ρ		$\gamma_{\rho+1}$		$\gamma_{\rho+2}$	
0	0	2	1.1313708498	9848	1.5084944665	3130	.1885618083	1641
2	2	4	.2310630260	3216	.7764501987	8171	.1640596504	3994
4	4	6	.1655774236	3764	.5980964283	2348	.1353056336	0524
6	6	8	.1356417322	8876	.5047289526	5037	.1175040789	8690
8	8	10	.1176233710	8966	.4448575637	8829	.1052201359	6643
10	10	12	.1052740554	5961	.4022649550	3295	.0961088994	4619
12	12	14	.0961371847	1805	.3699645475	8905	.0890098109	6527
14	14	16	.0890262326	7909	.3443801010	9916	.0832788926	7387
16	16	18	.0832891554	4772	.3234648466	3407	.0785273721	5678
18	18	20	.0785341551	7057	.3059512503	0766	.0745051302	8236
0	0	1	1.2000000000	0000	.9333333333	3333	−.1333333333	3333
1	1	3	.3164089278	5952	.9620520412	2313	.1856406460	5510
3	3	5	.1904294381	1627	.6698545132	1504	.1477503885	3051
5	5	7	.1484060475	2521	.5454750482	2134	.1254855713	8305
7	7	9	.1256780938	1192	.4719596648	4829	.1108596192	1061
9	9	11	.1109380083	2887	.4219542418	8812	.1003573304	9381
11	11	13	.1003958016	6826	.3851010847	3850	.0923560838	1042
13	13	15	.0923774080	0326	.3564848982	4339	.0860018352	4021
15	15	17	.0860147146	7456	.3334311090	0651	.0807987351	3941
17	17	19	.0808070274	1129	.3143424984	0354	.0764371100	3120
18	19	20	−.0188722146	3033	.1514654391	3771	.0938807984	1044
4	5	6	−.0355555346	9219	.2877732066	5162	.1746258586	0734
8	9	10	−.0270425254	7563	.2177871865	2965	.1338106592	8274
14	15	16	−.0211683402	9753	.1700363973	6522	.1051652505	1748

TABLE 8b

Coefficients (for $A(x)$) for evaluating $\int_{\tau_1 h}^{\tau_2 h} x^{1/2} A(x)\,dx$

ρ	τ_1	τ_2	γ_ρ	$\gamma_{\rho+1}$	$\gamma_{\rho+2}$
0	0	2	.107749604 7522	1.292995257 0268	.484873221 3851
2	2	4	.475357664 7353	2.302887371 0031	.669470214 4308
4	4	6	.668337573 0153	2.978426384 6878	.817861680 0962
6	6	8	.817469910 4893	3.525863635 0391	.943652148 6519
8	8	10	.943464314 0503	3.998763383 4285	1.054678704 9974
10	10	12	1.054571955 9477	4.421251702 6591	1.155138194 7060
12	12	14	1.155070784 9590	4.806689981 9729	1.247561921 8562
14	14	16	1.247516176 9228	5.163403677 6562	1.333611202 1976
16	16	18	1.333578490 1361	5.496998386 8340	1.414444701 7946
18	18	20	1.414420358 5575	5.811462624 8711	1.490908171 1344
0	0	1	.209523809 5238	.514285714 2857	−.057142857 1429
1	1	3	.341757835 0590	1.873406268 0472	.582270845 3648
3	3	5	.579765105 6142	2.662464470 3295	.747228733 9179
5	5	7	.746600496 8708	3.263709770 8386	.882969258 9261
7	7	9	.882705350 9161	3.769759930 3210	1.000695267 1281
9	9	11	1.000555931 8310	4.215314705 1390	1.106044492 3030
11	11	13	1.105960561 3636	4.617999530 8364	1.202235832 5482
13	13	15	1.202180719 7732	4.988239743 9456	1.291301944 3341
15	15	17	1.291263478 5652	5.332812227 2107	1.374621255 8168
17	17	19	1.374593160 0054	5.656417583 2367	1.453178784 6063
18	19	20	−.367983987 0759	2.934411591 0762	1.849331844 4789
4	5	6	−.195393566 2677	1.545208888 2560	.994583724 1451
8	9	10	−.256832789 4162	2.041069849 9891	1.297614007 2163
14	15	16	−.328075125 2726	2.613982682 6926	1.650925647 1724

TABLE 9a

Coefficients (for $A(x^2)$) for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x^2)\,dx$

p	r_1	r_2	γ'_p	γ'_{p+1}	γ'_{p+2}
0	0	2	1.2570787221 0942	1.3408839702 5005	.2304644323 8673
2	2	4	.2337416570 1354	.7718582599 5649	.1659729582 8378
4	4	6	.1662500838 4864	.5968734097 5803	.1358559919 5969
6	6	8	.1359207681 9108	.5042080856 3271	.1177459101 0224
8	8	10	.1177692905 6752	.4445810847 7760	.1053506954 9926
10	10	12	.1053613424 0978	.4020979713 0220	.0961885962 2678
12	12	14	.0961942036 2452	.3698547333 9882	.0890626062 4904
14	14	16	.0890658694 7398	.3443033847 2195	.0833159722 5619
16	16	18	.0833180147 2596	.3234087771 7921	.0785545823 3340
18	18	20	.0785559336 7650	.3059088101 4226	.0745257919 4183
0	0	1	1.5555555555 5556	.4592592592 5926	−.0148148148 1481
1	1	3	.3250988610 3139	.9481481481 4815	.1908546059 5822
3	3	5	.1916486313 0101	.6676870586 6439	.1486086498 9643
5	5	7	.1488227697 7184	.5447057148 4293	.1258381825 1482
7	7	9	.1258756714 3955	.4715877540 1980	.1110339524 1146
9	9	11	.1110493834 1746	.4217420988 6224	.1004580984 3110
11	11	13	.1004657121 1712	.3849668566 7669	.0924204014 2337
13	13	15	.0924246344 1675	.3563937024 1044	.0860458046 5967
15	15	17	.0860483661 3397	.3333658455 7008	.0808303471 1644
17	17	19	.0808319983 5781	.3142939062 9138	.0764607311 9683
18	19	20	−.0204117899 0198	.1544656371 0296	.0924201757 1683
4	5	6	−.0476838734 9549	.3098247317 4854	.1647026723 1373
8	9	10	−.0318765941 6382	.2269464745 7043	.1294854399 3015
14	15	16	−.0233752002 2216	.1743077391 5484	.1031007686 5249

TABLE 9b

Coefficients (for $A(x^3)$) for evaluating $\int_{r_1 h}^{r_2 h} x^{1/3} A(x^3)\,dx$

p	r_1	r_2	γ'_p		γ'_{p+1}		γ'_{p+2}	
0	0	2	−.097954186	1384	1.567266978	2143	.416305291	0882
2	2	4	.452216167	1807	2.342558509	6683	.652940573	3203
4	4	6	.658392304	3145	2.996508691	4166	.809724642	0683
6	6	8	.811652939	7074	3.536721980	4986	.938610773	9742
8	8	10	.939541877	5276	4.006195368	4189	1.051169156	5297
10	10	12	1.051699994	4541	4.426745889	8643	1.152515968	9944
12	12	14	1.152851749	2931	4.810963680	2923	1.245507259	2027
14	14	16	1.245735353	4039	5.166850432	8540	1.331945270	5186
16	16	18	1.332108486	3337	5.499854394	2216	1.413058698	2095
18	18	20	1.413180212	1146	5.813879320	5033	1.489731621	9450
0	0	1	.354978354	9784	.320346320	3463	−.008658008	6580
1	1	3	.294388630	1214	1.949196995	9475	.553849322	4022
3	3	5	.565463128	0651	2.687890208	1945	.736104973	6020
5	5	7	.739174086	4688	3.277420066	9653	.876685373	2014
7	7	9	.877989983	9428	3.778635915	2120	.996534649	2105
9	9	11	.997226622	2264	4.221656247	2429	1.103032259	8036
11	11	13	1.103450030	8375	4.622819749	4466	1.199926144	4642
13	13	15	1.200200826	2146	4.992062986	6795	1.289458595	1588
15	15	17	1.289650462	6946	5.335940500	4142	1.373105998	4838
17	17	19	1.373246215	6420	5.659038664	1600	1.451904648	0463
18	19	20	−.398056079	3024	2.993013616	9536	1.820801910	8280
4	5	6	−.262478806	2538	1.667182051	8672	.939695800	5201
8	9	10	−.302911628	3557	2.128377123	7693	1.256385572	3756
14	15	16	−.362353321	9191	2.680327579	4278	1.618858947	0838

TABLE 10a

Coefficients (for $A(x)$) for evaluating $\int_{r_1 h}^{r_2 h} x^{-1/2} A(x)\,dx$

ρ	r_1	r_2	γ_ρ	$\gamma_{\rho+1}$	$\gamma_{\rho+2}$	$\gamma_{\rho+3}$	$\gamma_{\rho+4}$
−1	0	2	−.0329234903 4096	1.2271482763 4491	1.4187031292 3777	.2125061649 2802	+.0029930445 7645
1	2	4	−.0087403039 9604	.2616137044 5934	.7372399874 7644	.1857892537 5312	−.0043297664 3904
3	4	6	−.0060061589 3628	.1875927665 6118	.5680873531 7056	.1533023908 8560	−.0039968661 1469
5	6	8	−.0048167473 7336	.1537022161 9663	.4794479851 6693	.1331515517 2361	−.0036102417 8779
7	8	10	−.0041239802 0249	.1332935101 2193	.4225910279 0640	.1192387114 4334	−.0032981984 2480
9	10	12	−.0036598242 6662	.1193028854 0885	.3821374107 8493	.1089167951 6098	−.0030493571 4939
11	12	14	−.0033220334 4044	.1089504623 6567	.3514569152 8881	.1008733763 8463	−.0028471773 2631
13	14	16	−.0030624160 1758	.1008929236 1087	.3271545244 0927	.0943796373 2861	−.0026794428 7905
15	16	18	−.0028550081 5832	.0943918537 5680	.3072868006 5676	.0889954018 1745	−.0025376738 3412
17	18	20	−.0026844607 2208	.0890034762 2982	.2906500514 6240	.0844374076 8345	−.0024159388 9300
−1	0	1	−.0455026455 0265	1.3121693121 6931	.8698412698 4127	−.1608465608 4656	+.0243386243 3862
0	1	3	−.0124341561 6106	.3576959167 7061	.9127960114 5625	.2100283634 9985	−.0039845204 2789
2	3	5	−.0070256187 0255	.2157063158 3900	.6361775922 6512	.1673760720 7535	−.0042000216 1410
4	5	7	−.0053182836 5060	.1681557623 7934	.5181356055 6252	.1421884467 4068	−.0037948639 0234
6	7	9	−.0044314144 3983	.1424175201 6034	.4483298724 4198	.1256265827 3727	−.0034451830 2894
8	9	11	−.0038716212 4364	.1257198743 4639	.4008383712 9740	.1137299585 9725	−.0031670022 8660
10	11	13	−.0034788819 0576	.1137757482 8705	.3658345363 1668	.1046648684 2072	−.0029433009 0151
12	13	15	−.0031844119 8279	.1046902506 9078	.3386528420 7756	.0974650674 4046	−.0027596067 3915
14	15	17	−.0029533174 7899	.0974803984 4161	.3167539625 7932	.0915692466 0862	−.0026057313 3007
16	17	19	−.0027658303 6463	.0915791175 3199	.2986212102 2921	.0866267374 7627	−.0024745990 2681
16	19	20	−.0059681396 6027	.0333005998 8435	−.0829651763 2179	.2036221215 0863	.0784846175 0688
2	5	6	−.0112053997 7638	.0625454082 7184	−.1559593608 4940	.3857662332 5608	.1456966496 6465
6	9	10	−.0085397589 8394	.0476565644 7278	−.1187736649 9035	.2924388080 7650	.1117733717 6176
12	15	16	−.0066919325 1814	.0373405171 0829	−.0930382965 1354	.2285224885 4495	.0879005309 6361
0	0	2	1.0625308246 4010	1.7479380326 4738	−.1167287384 8159	.1676104962 8126	−.0329234903 4096
0	0	1	1.0846560846 5608	1.3248677248 6772	−.6158730158 7302	.2518518518 5185	−.0455026455 0265

TABLE 10b

Coefficients (for $A(x)$) for evaluating $\int_{r_1 h}^{r_2 h} x^{1/2} A(x)\,dx$

p	r_1	r_2	γ_p	γ_{p+1}	γ_{p+2}	γ_{p+3}	γ_{p+4}
−1	0	2	+.001632569 7690	.125163682 2880	1.230957605 8058	.550176012 1440	−.022311786 8426
1	2	4	−.012673895 1896	.538961773 6926	2.188118415 2684	.758891379 7864	−.025582423 3887
3	4	6	−.019811792 2890	.757544328 1653	2.829676872 9718	.926987607 2342	−.029771378 2830
5	6	8	−.025160762 6842	.926521400 4494	3.349673741 2641	1.069520517 0585	−.033569201 9075
7	8	10	−.029605079 4024	1.069296952 9152	3.798895943 2481	1.195335986 3731	−.037017400 6578
9	10	12	−.033483435 2648	1.195208921 5496	4.200241417 4426	1.309181609 3929	−.040186659 8074
11	12	14	−.036966322 5618	1.309101367 2558	4.566396170 4530	1.413923088 2526	−.043131614 6115
13	14	16	−.040153334 0282	1.413868633 7509	4.905266311 3409	1.511441900 4565	−.045892454 7436
15	16	18	−.043108447 2412	1.511402959 5745	5.222175661 9663	1.603050532 1799	−.048499127 7147
17	18	20	−.045875577 5756	1.603021553 3376	5.520912505 9845	1.689707134 8698	−.050974462 0532
−1	0	1	−.017027417 0274	.250120250 1202	.494660894 6609	−.071572871 5729	+.010485810 4858
0	1	3	−.007524000 6587	.387791919 4055	1.780448018 9602	.660181093 1344	−.023462082 3703
2	3	5	−.016572546 6380	.657204184 4457	2.529582513 6630	.846965597 3084	−.027721438 9175
4	5	7	−.022632966 4153	.846218139 1483	3.100654642 4977	1.000758454 4364	−.031718743 0317
6	7	9	−.027468918 4368	1.000444368 3137	3.581356388 7488	1.134160971 8268	−.035332262 0873
8	9	11	−.031602033 8313	1.133995124 6464	4.004609329 6807	1.253545800 0986	−.038633091 3214
10	11	13	−.035266984 6572	1.253445893 5560	4.387145442 2025	1.362555951 8678	−.041684378 2208
12	13	15	−.038592215 7173	1.362490346 6754	4.738864157 5430	1.463493099 3021	−.044532979 7502
14	15	17	−.041656777 9159	1.463447309 4641	5.066201402 0092	1.557918525 1865	−.047213497 1513
16	17	19	−.044513306 1602	1.557885079 3486	5.373621662 1686	1.646948093 3539	−.049752000 8625
16	19	20	−.116685500 0067	.650891259 8039	−1.620544766 4471	3.953601370 4339	1.548497084 6952
2	5	6	−.062170680 1761	.346675747 3088	−.862396727 1375	2.087870688 7736	.834420017 3646
6	9	10	−.081555639 5510	.454864293 8991	−1.132091833 8073	2.753217615 2782	1.087416631 9702
12	15	16	−.104066814 5855	.580481219 8790	−1.445117897 3965	3.522891825 6455	1.382644871 0500
0	0	2	.133326531 1328	1.214631908 1161	+.566501709 8337	−.030474635 6875	.001632569 7690
0	0	1	.164983164 9832	.664935064 9351	−.241847041 8470	.095622895 6229	−.017027417 0274

TABLE 11

Constants for the estimation of error

ρ	n_1	n_2	$\alpha = -\frac{1}{2}$		$\alpha = \frac{1}{2}$	
			K_1	K_2	K_1	K_2
			$p = 2$			
0	0	2	.216	.503	−.144	−.823
4	4	6	.0121	.121	−.0598	−1.79
8	8	10	.00496	.0893	−.0445	−2.40
18	18	20	.00161	.0612	−.0306	−3.49
0	0	1	.419		.165	
4	5	6	−.106		−.588	
8	9	10	−.0810		−.772	
18	19	20	−.0566		−1.105	
			$p = 4$			
0	0	2	3.20	32.3	.315	2.19
−1	0	2	−.751	−3.24	.511	3.96
3	4	6	−.0430	−.863	.213	8.53
7	8	10	−.0177	−.637	.159	11.43
17	18	20	−.00575	−.437	.109	16.61
0	0	1	3.96		1.43	
2	5	6	−.954		−5.31	
6	9	10	−.728		−6.96	
16	19	20	−.509		−9.95	
−1	0	1	−1.501		−.616	
3	5	6	.390		2.15	
7	9	10	.297		2.83	
17	19	20	.207		4.05	

TABLE 12

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} [A(x) \log x + B(x)]\, dx$

p	r_1	r_2	c_p	c_{p+1}	c_{p+2}	c_{p+3}	c_{p+4}	c_{p+5}
1	0	1	4.39444 96934	−11.39989 11269	15.74964 18868	−10.37740 91666	2.63320 87133	←
1	1	2	.30835 63122	1.06758 76074	−.63623 40289	.33627 99867	−.07598 98774	(This first group
1	2	3	−.01513 50788	.43324 44247	.70774 38653	−.15468 06895	.02882 74783	assumes that
1	3	4	.00664 16933	−.06663 90095	.57673 35354	.51988 31845	−.03661 94037	log x occurs
1	4	5	−.00918 12721	.07547 15778	−.25466 04340	.81963 12231	.36873 89052	only as (x log
1	0	5	4.68513 13481	−9.89022 65266	16.14322 48246	−8.85629 54618	2.91816 58157	x)· analytic
								function)
1	1	2	.26120 69063	1.41104 49018	−1.55813 30185	1.51272 72490	−.80126 95809	.17442 35423
1	2	3	−.00636 50499	.36935 96246	.87922 17567	−.37350 58517	.16373 31845	−.03244 36642
1	3	4	.00155 22680	−.02956 53667	.47722 14435	.64687 18562	−.11490 79161	.01882 77151
1	4	5	−.00104 13216	.01617 65492	−.09550 22859	.61652 74453	.49395 23784	−.03011 27654
1	1	5	.25535 28028	1.76701 57088	−.29719 21042	2.40262 06988	−.25849 19341	.13069 48279
1	0	1	12.96004 65124	−73.79551 72935	183.23033 21174	−224.10169 10489	134.39422 23575	−31.68739 26449
1	5	6	.00183 13157	−.02579 19022	.12554 00878	−.32351 53403	.86442 14475	.35751 43915
1	0	6	13.21723 06309	−72.05429 34869	183.05868 01010	−222.02258 56904	135.00015 18709	−31.19918 34255
5	5	6	.30854 80701	1.10471 37171	−.79636 12091	.59198 72901	−.25648 61720	.04759 83038
5	6	7	−.01321 35314	.41293 63222	.77214 11394	−.24523 93708	.08837 76916	−.01500 22510
5	7	8	.00431 87821	−.04700 04545	.51976 77392	.59617 83551	−.08525 20203	.01198 75984
5	8	9	−.00355 03311	.03188 73472	−.13366 03855	.66182 70513	.46752 93207	−.02403 30026
5	9	10	.00728 53348	−.05978 28337	.20780 91016	−.42091 69836	.92111 13234	.34449 40575
5	5	10	.30338 83244	1.44275 40984	.56969 63856	1.18383 63421	1.13528 01434	.36504 47061
10	10	11	.31825 77916	1.05110 19210	−.67871 24272	.46361 45061	−.18678 83726	.03252 65811
10	11	12	−.01547 05038	.42538 77508	.74483 68113	−.21546 52342	.07222 15503	−.01151 03744
10	12	13	.00553 86787	−.05372 60040	.53450 74263	.58011 36249	−.07653 89062	.01010 51803
10	13	14	−.00491 16811	.03938 86348	−.15009 22743	.67972 84924	.45782 37047	−.02193 68765
10	14	15	.01075 82272	−.07891 01388	.24969 10871	−.46652 80335	.94583 22251	.33915 66329
10	10	15	.31417 25126	1.38324 21638	.70023 06230	1.04146 33557	1.21255 02014	.34834 11435
15	15	16	.32187 71187	1.03189 02007	−.63801 85452	.42060 99114	−.16411 05833	.02775 18977
15	16	17	−.01641 93182	.43042 35422	.73417 12747	−.20419 52283	.06627 90548	−.01025 93252
15	17	18	.00610 44216	−.05672 83471	.54086 56178	.57339 56794	−.07299 69320	.00935 95603
15	18	19	−.00559 75881	.04302 83405	−.15779 95806	.68787 12459	.45353 08281	−.02103 32459
15	19	20	.01263 84416	−.08888 65006	.27081 50360	−.48884 38541	.95759 63945	.33668 04826
15	15	20	.31860 30757	1.35972 72356	.75003 38028	.98883 77543	1.24029 87621	.34249 93695

TABLE 13

Coefficients for evaluating $\int_{r_1 h}^{r_2 h} x[A(x) \log x + B(x)]\, dx$

ρ	r_1	r_2	c_ρ	$c_{\rho+1}$	$c_{\rho+2}$	$c_{\rho+3}$	$c_{\rho+4}$	$c_{\rho+5}$
				$r = 1 = \rho$ to $r = 6$				
1	1	2	.31985 83337	1.06904 51984	−.73371 02295	.52948 32538	−.22349 21106	.04045 15827
1	2	3	−.01532 59872	.42029 51558	.75824 15542	−.23069 73298	.08046 80014	−.01325 66124
1	3	4	.00537 09023	−.05093 90581	.52752 03239	.58788 58565	−.08068 90295	.01097 46606
1	4	5	−.00468 87411	.03636 98192	−.14270 60117	.67161 75881	.46211 30437	−.02282 80722
1	1	5	.30521 45076	1.47477 11154	.40934 56369	1.55828 93686	.23839 99050	.01534 15586
1	0	1	2.45916 60147	−5.35319 76502	8.35585 23292	−7.44107 31602	3.50446 64227	−.67972 78624
1	5	6	.01016 77397	−.07155 67713	.23195 92507	−.44724 06826	.93571 48687	.34124 27396
1	0	6	2.77454 82620	−3.94998 33062	8.99715 72168	−6.33002 44742	4.67858 11964	−.32314 35642
				$r = 5 = \rho$ to $r = 10$				
5	5	6	.32390 44767	1.02226 37731	−.61918 20256	.40180 45254	−.15459 17417	.02580 58628
5	6	7	−.01697 52038	.43309 13008	.72890 51793	−.19890 02591	.06358 34358	−.00970 56890
5	7	8	.00645 30827	−.05841 61493	.54422 03692	.57000 41026	−.07126 29124	.00900 22289
5	8	9	−.00604 32352	.04520 10608	−.16214 21732	.69228 03179	.45126 91821	−.02056 60158
5	9	10	.01392 71860	−.09520 72881	.28350 58472	−.50177 32982	.96424 57679	.33530 41343
5	5	10	.32126 63064	1.34693 26975	.77530 71968	.96341 53886	1.25324 37318	.33984 05211
				$r = 10 = \rho$ to $r = 15$				
10	10	11	.32633 35382	1.00900 77501	−.59095 27150	.37216 14675	−.13916 08332	.02261 09689
10	11	12	−.01767 94764	.43689 51047	.72085 86003	−.19049 01706	.05922 07645	−.00880 48718
10	12	13	.00690 93698	−.06086 26661	.54937 11177	.56463 88778	−.06848 68268	.00843 01587
10	13	14	−.00663 76748	.04837 08562	−.16879 14465	.69918 84394	.44770 18132	−.01983 20272
10	14	15	.01566 51927	−.10443 48472	.30280 63457	−.52178 31646	.97456 24385	.33318 41496
10	10	15	.32459 09496	1.32897 61978	.81329 19022	.92371 54495	1.27383 73562	.33558 83781
				$r = 15 = \rho$ to $r = 20$				
15	15	16	.32736 11436	1.00365 25992	−.57986 40585	.36073 58843	−.13329 52792	.02140 97320
15	16	17	−.01798 42426	.43847 92507	.71758 50196	−.18712 26484	.05749 43244	−.00845 17098
15	17	18	.00711 22209	−.06191 48516	.55154 18086	.56240 89103	−.06734 48634	.00819 67793
15	18	19	−.00690 94896	.04977 82614	−.17169 08883	.70216 36475	.44617 96659	−.01952 12022
15	19	20	.01648 25826	−.10866 06712	.31150 15161	−.53069 66822	.97911 89033	.33225 43671
15	15	20	.32606 22149	1.32133 45883	.82907 33976	.90748 91115	1.28215 27510	.33388 79663

TABLE 14

Inverse matrices for $A(x) \log x + B(x)$

			$r = \rho = 1$ to $r = 6$					
−845.4	62277	599761	−12589.9	11408	30899	−10565.1	53959	06207
521.2	27469	866092	8074.0	21030	97579	7125.1	08459	39675
−125.3	50731	890576	−2000.4	15777	16675	−1842.3	08825	17171
			$r = \rho = 5$ to $r = 10$					
−56621.	07468	72845	−318613.	09503	40962	−110251.	94283	94330
31305.	56227	83484	180170.	01705	17434	63901.	39224	87654
−6879.	69942	05226	−40369.	08130	33728	−14646.	32311	38439
			$r = \rho = 10$ to $r = 15$					
−76055	7.6181	15222	−248799	9.9798	10215	−50584	3.7252	45206
40442	6.7446	29950	134224	3.5148	14174	27707	8.3028	26441
−8581	0.0856	31170	−28855	9.8677	59825	−6043	3.5403	81232
			$r = \rho = 15$ to $r = 20$					
−41240	21.809	63656	−95478	78.688	27772	−13776	67.532	43689
21556	67.931	37710	50440	59.815	81456	7358	67.270	02698
−4501	29.455	654850	−10637	48.537	204500	−1568	42.008	941664

These correspond to the following values of the original determinant (assumed to be exact):

$10^{-4} \cdot .27329\ 86064\ 01449$ $\quad$ $10^{-13} \cdot 1.1438\ 57228\ 31$

$10^{-13} \cdot 1.2758\ 69611\ 69$ $\quad$ $10^{-13} \cdot .53662\ 370107$

A NEW INTEGRATING PROCEDURE OF HIGH ACCURACY

MARK LOTKIN

1. Introduction. While there are presently available many methods for the numerical solution of the initial value problem for systems of ordinary differential equations, the procedure described in the following has a higher accuracy than most of these other methods, and is still relatively easy to apply.

This procedure may be summarized as follows: For the system of differential equations

$$y_i' = f_i(x, y_1, \cdots, y_n), \qquad y_i(x_0) = y_{i0} \qquad \text{for } i = 1, 2, \cdots, n$$

let

$$|\partial f_i^{(k)}/\partial y_\nu| \leq L \text{ for } k = 0, 1, 2, \text{ all } i, \nu,$$

$$|d^k f_i/dx^k| \leq M \text{ for } k = 0, 1, \cdots, 6, \text{ all } i,$$

in the region $|x - x_0| \leq h$, $|y_i - y_{i0}| \leq 2hM$, with L, M denoting absolute positive constants. Then the sequence of functions $y_{i,j}(x)$ defined by the iterative procedure

$$y_{i,j+1}(x) = y_{i0} + \tfrac{1}{2}x(f_{i0} + f_{ij}) + \tfrac{1}{10}x^2(f_{i0}' - f_{ij}') + \tfrac{1}{120}x^3(f_{i0}'' + f_{ij}''),$$

$$f_{i0} \equiv f_i(x_0, y_{10}, \cdots, y_{n0}), \qquad f_{ij} \equiv f_i(x, y_{1j}(x) \cdots, y_{nj}(x)),$$

differs from the solution of the system by a term of the type $O(h^7)$, provided h is made sufficiently small and j sufficiently large.

It is thus seen that this integration scheme is equivalent to a local approximation to the solution by a sixth-degree polynomial.

The theorem stated above is first proved for a single differential equation, and is then extended to systems. A few simple examples show the estimates for the truncation errors to be very close to the actual errors.

The iterative method described here is superior to such methods as Adams', Moulton's, etc. in that the integrating procedure is the same for early steps as for later steps, and in that a reduction in the size of the integration interval does not necessitate any interpolation.

2. The quadrature formula. Let us first consider the simple case

$$y' = f(x), \qquad y(x_0) = y_0, \tag{1}$$

and calculate y at $x_1 = x_0 + h$. It will be shown that

$$y_1 = y_0 + I, I = \int_{x_0}^{x_0+h} f(t)\,dt = \tfrac{1}{2}h(f_0 + f_1) + \tfrac{1}{10}h^2(f_0' - f_1') + \tfrac{1}{120}h^3(f_0'' + f_1'') + R, \tag{2}$$

where

$$f_i^{(k)} = (\partial^k f/\partial x^k)_{x=x_i}, \qquad i = 0, 1; k = 0, 1, 2,$$

and

$$R = -(h^7/201{,}600)(f_0^{(6)} + f_1^{(6)}) + O(h^9). \tag{3}$$

Formula (2) represents an application of the well-known Euler-MacLaurin formula.

To prove Eq. (2) we expand $f(t)$ first in a Taylor series about x_0 and integrate between x_0 and x_1, obtaining

$$I = hf_0 + (h^2/2)f_0' + \cdots + (h^7/7!)f_0^{(6)} + O_1(h^8).$$

Next we expand $f(t)$ similarly about x_1, and integrate:

$$I = hf_1 - (h^2/2)f_1' + \cdots + (h^7/7!)f_1^{(6)} + O_2(h^8).$$

Averaging we find that

$$I = \tfrac{1}{2}h(f_0 + f_1) + \tfrac{1}{4}h^2(f_0' - f_1') + \tfrac{1}{2}\sum_{i=3}^{7}(h^i/i!)[f_0^{(i-1)} + (-1)^{i-1}f_1^{(i-1)}] + O_3(h^8).$$

We split up the term $(h^2/4)(f_0' - f_1')$ into $[(h^2/10) + (3h^2/20)](f_0' - f_1')$, getting

$$I = \tfrac{1}{2}h(f_0 + f_1) + \tfrac{1}{10}h^2(f_0' - f_1') + \tfrac{3}{20}h^2(f_0' - f_1') + \tfrac{1}{2}\sum_{i=3}^{7}(h^i/i!)[\cdots] + O_3(h^8)$$

Now f_0' is expanded about x_1, f_1' about x_0, and $f_0' - f_1'$ is formed:

$$2(f_0' - f_1') = -\sum_{i=1}^{6}(h^i/i!)[f_0^{(i+1)} + (-1)^{i+1}f_1^{(i+1)}] + O_4(h^8).$$

Consequently, inserting $(3h^2/20)(f_0' - f_1')$ into the last expression for I, and combining terms:

$$I = (h/2)(f_0 + f_1) + (h^2/10)(f_0' - f_1') + (h^3/120)(f_0'' + f_1'') + R,$$

$$R = -(h^4/60)(f_0^{(3)} - f_1^{(3)}) - (h^5/120)(f_0^{(4)} + f_1^{(4)}) - (7h^6/2880)(f_0^{(5)} - f_1^{(5)})$$
$$- (53h^7/100{,}800)(f_0^{(6)} + f_1^{(6)}) + O_5(h^8).$$

In the next step we expand $f_0^{(3)}$ about x_1, $f_1^{(3)}$ about x_0, form $f_0^{(3)} - f_1^{(3)}$, and get

$$2(f_0^{(3)} - f_1^{(3)}) = -\sum_{i=1}^{3}(h^i/i!)[f_0^{(i+3)} + (-1)^{i+3}f_1^{(i+3)}) + O_6(h^8).$$

The introduction of $(h^4/60)(f_0^{(3)} - f_1^{(3)})$ into the last equation for I and combination of terms now leads to

$$R = (h^6/576)(f_0^{(5)} - f_1^{(5)}) + (29h^7/33{,}600)(f_0^{(6)} + f_1^{(6)}) + O_7(h^8).$$

Finally, from

$$f_0^{(5)} = f_1^{(5)} - hf_1^{(6)} + \cdots, \quad f_1^{(5)} = f_0^{(5)} + hf_0^{(6)} + \cdots,$$

it follows again that

$$2(f_0^{(5)} - f_1^{(5)}) = -h(f_0^{(6)} + f_1^{(6)}) - \cdots,$$

whence

$$R = -(h^7/201{,}600)(f_0^{(6)} + f_1^{(6)}) + O(h^8),$$

as claimed in Eq. (3). We may thus infer that

$$R = -(h^7/100{,}800)f^{(6)}(x_0 + \theta h), \qquad 0 < \theta < 1. \tag{4}$$

3. Example. $I = \int_0^{\pi/6} \cos t\, dt$. Using formula (2) one finds

$$I = (\pi/12)(1 + \sqrt{3}/2) + (\pi^2/720) - (\pi^3/25{,}920)(1 + \sqrt{3}/2) = .499\ 999\ 899. \tag{5}$$

The exact value is $I = 0.5$, so that the error is $1.01 \cdot 10^{-7}$. By Eq. (4),

$$|R| \leq (\pi/6)^7/10^5 < 1.08 \cdot 10^{-7},$$

which is very close indeed to the exact value.

Incidentally, the example shows that the algebraic equation

$$(1 + \sqrt{3}/2)x(1 - x^2/2160) + x^2/60 = 6$$

has as a root $x = 3.1415933$, which differs from π by about $6 \cdot 10^{-7}$.

4. The iteration method. Let us now consider the general case

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{6}$$

and assume that in the rectangle $|x - x_0| \leq h$, $|y - y_0| \leq 2hM$, $|\partial f^{(k)}/\partial y| \leq L$ for $k = 0, 1, 2$, and $|\partial^k f/dx^k| \leq M$ for $k = 0, 1, \cdots, 6$, with L and M denoting positive constants that are independent of x, y, k. In this case we approximate the solution $y(x)$ in $|x - x_0| < h$ by a succession $y_j = y_j(x)$ of iterations; for $x = x_1$, in particular, using the notation

$$f_0 \equiv f(x_0, y_0), \qquad f_j \equiv f(x_1, y_j(x_1))$$

we put

$$y_{j+1,1} = y_0 + I_j \quad \text{for} \quad j \geq 0, \tag{7}$$

$$I_j = \tfrac{1}{2}h(f_0 + f_j) + \tfrac{1}{10}h^2(f_0' - f_j') + \tfrac{1}{120}h^3(f_0'' + f_j''). \tag{8}$$

The iterations may be started with $y_0(x_1)$ obtained from the expansion

$$y_0(x_1) \equiv y_{01} = y_0 + hf_0 + (\tfrac{1}{2}h^2)f_0' + (\tfrac{1}{6}h^3)f_0''. \tag{9}$$

It will in the following be shown that the local trunction error of this procedure approaches zero as h^7. To prove this we define the local errors:

$$t_j = y_{j,1} - y_1, \; t_j^{(k)} = f_j^{(k)} - f_1^{(k)}, \qquad k = 0, 1, 2,$$

with $f_j^{(0)} = f_j$, $f_1^{(0)} = f_1$. By subtraction, on the basis of Eqs. (7), (8), and (2),

$$t_{j+1} = (\tfrac{1}{2}h)t_j^{(0)} - (\tfrac{1}{10}h^2)t_j^{(1)} + (\tfrac{1}{120}h^3)t_j^{(2)} - R.$$

However, by the mean value theorem,

$$t_j^{(k)} = f^{(k)}(x_1, y_{j,1}) - f^{(k)}(x_1, y_1) = f_y^{(k)}(x_1, y_1 + \theta^{(k)}t_j)t_j \equiv f_{yj}^{(k)}t_j.$$

Consequently,

$$t_{j+1} = hC_jt_j - R,$$
$$C_j = (\tfrac{1}{2})f_{yj}^{(0)} - (\tfrac{1}{10}h)f_{yj}' + (\tfrac{1}{120}h^2)f_{yj}''.$$

Since, then,

$$t_j = hC_{j-1}t_{j-1} - R,$$

it follows that

$$t_{j+1} = h^2C_jC_{j-1}t_{j-1} - (1 + hC_j)R,$$

or, after j such substitutions,

$$\begin{aligned} t_{j+1} &= h^{j+1}C_jC_{j-1}\cdots C_1C_0t_0 \\ &\quad - (1 + hC_j + h^2C_jC_{j-1} + \cdots + h^jC_j\cdots C_1)R. \end{aligned} \tag{10}$$

We shall prove next that $|y_{j+1,1} - y_0| < 2hM$ for all $j \geq 0$. First, on the basis of our assumptions we may deduce from Eq. (9) that

$$\begin{aligned} |y_{01} - y_0| &\leq h[|f_0| + (\tfrac{1}{2}h)|f_0'| + (\tfrac{1}{6}h^2)|f_0''|] \\ &\leq hM[1 + (\tfrac{1}{2}h) + (\tfrac{1}{6}h^2)] < 2hM \end{aligned}$$

provided $h < 1$.

Assuming, inductively, that $|y_{j1} - y_0| < 2hM$, we may infer that $|f_j^{(k)}| \leq M$, so that, by Eqs. (7) and (8),

$$|y_{j+1,1} - y_0| \leq hM[1 + (\tfrac{1}{2}h) + (\tfrac{1}{60}h^2)] < 2hM,$$

provided $h < 1$.

We may now conclude that the $|f_{ij}^{(k)}| \leq L$ for $k = 0, 1, 2$, and all $j \geq 0$. It follows that

$$|C_j| \leq (\tfrac{1}{2}L)[1 + (\tfrac{1}{2}h) + (\tfrac{1}{60}h^2)] < L \quad \text{for all} \quad j \geq 0. \tag{11}$$

The local truncation error can now be estimated. By definition, $t_0 = y_{01} - y_1$, where y_{01} is exhibited in Eq. (9) and

$$y_1 = y_0 + hf_0 + (\tfrac{1}{2}h^2)f_0' + (\tfrac{1}{6}h^3)f_0'' + (\tfrac{1}{24}h^4)f'''(x_0 + \theta'h).$$

Thus

$$|t_0| \leq Mh^4/24.$$

Let us now impose on h the further restriction that

$$N \equiv 1 - H - ch^2 > 0, \tag{12}$$

where $H = Lh$, $c = 24 \cdot 10^{-5}$. Since it was previously postulated that $h < 1$, we must limit ourselves to

$$h < h^* = \min(1, (L + c)^{-1}). \tag{13}$$

From the relationships (8), (11) and (4) it follows that

$$\begin{aligned} |t_{j+1}| &< H^{j+1}|t_0| + |R|(1 - H^{j+1})/(1 - H) \\ &\leq (\tfrac{1}{24})[H^{j+1}Mh^4 + cMh^7(1 - H^{j+1})/(1 - H)] \\ &= [\tfrac{1}{24}(1 - H)][cMh^7 + NMh^4H^{j+1}]. \end{aligned}$$

If condition (13) is satisfied, then certainly $H < 1$, so that a sufficiently large choice of j will guarantee

$$NMh^4H^{j+1} < cMh^7. \tag{14}$$

As a matter of fact, a short calculation shows that condition (14) is satisfied for all

$$j > j^* = [\log (N/ch^3)/\log (1/H)] - 1, \tag{15}$$

provided h satisfies (13).

Summarizing we may state that

$$| t_{j+1} | < 2Mh^7/10^3(1 - H) \tag{16}$$

if $h < h^*, j > j^*$. By a sufficient number of iterations the local truncation error may be reduced to the order h^7.

5. Systems of differential equations. If a system the differential equations

$$y_i' = f_i(x, y_1, \cdots, y_n), \qquad y_i(x_0) = y_{i0}, \tag{17}$$

$i = 1, 2, \cdots, n$, is given, then the iterations are defined by

$$y_{i,j+1,1} = y_{i0} + I_{ij}, \tag{18}$$

$$I_{ij} = \tfrac{1}{2}h(f_{i0} + f_{ij}) + \tfrac{1}{10}h^2(f'_{i0} - f'_{ij}) + \tfrac{1}{120}h^3(f''_{i0} + f''_{ij}), \tag{19}$$

with

$$f_{i0}^{(k)} \equiv f_i^{(k)}(x_0, y_{10}, \cdots, y_{n0})$$
$$f_{ij}^{(k)} \equiv f_i^{(k)}(x_1, y_{1j}(x_1), \cdots, y_{nj}(x_1)),$$

$i = 1, 2, \cdots, n, j \geq 0$.

The introduction of the vectors

$$y \equiv \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \quad f \equiv \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}, \quad y_j(x) \equiv \begin{bmatrix} y_{1j}(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{nj}(x) \end{bmatrix}, \quad f_j^{(k)} \equiv \begin{bmatrix} f_{1j}^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{nj}^{(k)} \end{bmatrix}, \text{ etc.,}$$

permits one to express Eqs. (17) in the form (6), and Eqs. (18) and (19) in the form (7), (8), respectively.

Applying the same procedure that was used in the previous section in estimating the truncation error it is found again that

$$t_{j+1} = hC_jt_j - R,$$

but now t_j, R denote vectors, and C_j is a matrix of n^2 elements. The estimates expressed in (11) and (15) also remain unchanged, but they must be interpreted now as holding for all components of C_j, t_{j+1}, respectively.

6. Examples. Let us first consider the equation

$$y' = x + y, \qquad y(0) = 0.$$

Here $f = x + y$, $L = 1$, $M = 1.11$ for $0 \leq x \leq 0.1$, $M = 1.23$ for $0 \leq x \leq 0.2$, etc. The integrated values of y, for one integrating step, are shown in Table 1, and are compared with the exact solution $y(x) = e^x - x - 1$.

As may be inferred from the last two columns of this table, the bounds for the errors are very good, in all cases exhibited. In this particular example the successive approximations are found to approach the final value from below.

TABLE 1

The equation $y' = x + y$, $\quad y(0) = 0$

h	y (integrated)	y (exact)	Error	Est. Error Bound
.1	.005 170 918 077	.005 170 918 076	$1 \cdot 10^{-12}$	$2.5 \cdot 10^{-12}$
.2	.021 402 758 4	.021 402 758 2	$2 \cdot 10^{-10}$	$3.9 \cdot 10^{-10}$
.5	.148 721 40	.148 721 27	$13 \cdot 10^{-8}$	$52. \ 10^{-8}$
1.0	.718 313	.718 282	$31 \cdot 10^{-6}$	

TABLE 2

h	y (integrated)	y (exact)	Error	Est. Error
.1	.104 668 651 9	.104 668 652 5	$6 \cdot 10^{-10}$	$24 \cdot 10^{-10}$
.2	.217 395 53	.217 395 56	$3 \cdot 10^{-8}$	$92 \cdot 10^{-8}$
.5	.588 70	.588 65	$5 \cdot 10^{-5}$	

As an example of a system of equations we take the non-linear set

$$y' = f(x, y, z) = z$$

$$z' = g(x, y, z) = -2x(z - x)^2 + 1$$

with the initial conditions $y(0) = 0$, $z(0) = 1$. Now

$$f' = g, f'' = -2(z - x)^2[1 - 4x^2(z - x)]$$

$$g' = f'', g'' = 24x(z - x)^3[1 - 2x^2(z - x)],$$

so that

$$y_{01} = h + (\tfrac{1}{2}h^2) - (\tfrac{1}{3}h^3), \qquad z_{01} = 1 + h - h^2,$$

$$y_{j+1,1} = (\tfrac{1}{2}h)(1 + f_j) + (\tfrac{1}{10}h^2)(1 - g_j) + (\tfrac{1}{120}h^3)(-2 + g'_j)$$

$$z_{j+1,1} = 1 + (\tfrac{1}{2}h)(1 + g_j) - (\tfrac{1}{10}h^2)(2 + g'_j) + (\tfrac{1}{120}h^3)g''.$$

Some of the results are given in Table 2, again for a single integrating step, and compared with the exact solution $y = \arctan x + (x^2/2)$.

The estimate of the error is based on values of 4 and 720 for L, M, respectively.

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ON THE CONVERGENCE OF NUMERICAL SOLUTIONS OF THE HEAT-FLOW EQUATION

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1. Introduction. Considerable discussion exists in the literature with regard to the replacement of the problem

$$u_{xx}(x, t) = u_t(x, t) \qquad (0 < x < 1, t > 0), \tag{1a}$$

$$u(0, t) = u(1, t) = 0 \qquad (t > 0), \tag{1b}$$

$$u(x, 0) = f(x) \qquad (0 < x < 1) \tag{1c}$$

by the problem

$$r(v_{\mu+1,\nu} - 2v_{\mu,\nu} + v_{\mu-1,\nu}) = v_{\mu,\nu+1} - v_{\mu,\nu}$$

$$(\mu = 1, 2, \cdots, M - 1, \quad \nu = 1, 2, \cdots), \tag{2a}$$

$$v_{0,\nu} = v_{M,\nu} = 0 \qquad (\nu = 1, 2, \cdots), \tag{2b}$$

$$v_{\mu,0} = f(\mu/M) \qquad (\mu = 1, 2, \cdots, M - 1), \tag{2c}$$

where $v_{\mu,\nu} \equiv v(x_\mu, t_\nu) \equiv v(\mu\Delta x, \nu\Delta t)$, and where the time and distance spacings Δt and $\Delta x = 1/M$ are related by the equation

$$\Delta t = r(\Delta x)^2 = r/M^2. \tag{3}$$

It is known [2] that the formulation (2) is *unstable* when $r > \frac{1}{2}$, that is, that the effects of numerical errors may be propagated in the t-direction with exponentially increasing magnitude in this case, whereas the formulation is stable when $r \leqq \frac{1}{2}$.

In the case when the strip $(0 < x < 1, t > 0)$ is replaced by the half-plane $t > 0$, it has been shown somewhat formally [1] that the solution of (2) *converges* to that of (1) as $M \to \infty$ when r is taken as $\frac{1}{2}$. Further, it has been commonly supposed that, for any fixed $r \leqq \frac{1}{2}$, the solution of (2) converges to the solution of (1) as the number M of subdivisions in the x-interval (0, 1) increases without limit, for any sufficiently regular function $f(x)$. However, no proof of this fact appears to exist in the literature.

In a recent paper [3] Leutert constructed a certain sequence of solutions, each of which satisfies (2a) and (2b), and showed that that sequence tends to the solution of (1) for *any* fixed positive value of r, as $M \to \infty$. However, in general, the successive solutions in that sequence (corresponding to successive refinements of the net) do not satisfy (2c) at mesh points along the initial line, but instead satisfy a different sequence of initial conditions which also tends to (1c) as $M \to \infty$, the determination of which sequence essentially depends upon advance knowledge of the solution of (2).

In the present paper, the *strict* convergence problem is considered, and convergence is established when $0 < r < \frac{1}{2}$ for any function $f(x)$ of a very general class.* Convergence when $r = \frac{1}{2}$ is established here only when $f(x)$ is rather severely restricted.

* Strict convergence was established by Leutert [3], in the special case $f(x) = 1$, when $0 < r \leqq 1/4$.

2. The convergence theorem. For any function $f(x)$ of bounded variation in (0, 1), the solution of the exact problem (1) is of the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{4}$$

where

$$a_n = 2\int_0^1 f(x) \sin n\pi x \, dx \qquad (n = 1, 2, \cdots), \tag{5}$$

whereas the solution of (2) is of the form

$$v_{\mu,r}^{(M)} = \sum_{n=1}^{M-1} b_n^{(M)} \left(1 - 4r \sin^2 \frac{n\pi}{2M}\right)^{\mu} \sin \frac{n\pi\mu}{M}, \tag{6}$$

where

$$b_n^{(M)} = \frac{2}{M} \sum_{k=1}^{M-1} f\left(\frac{k}{M}\right) \sin \frac{n\pi k}{M} \qquad (n = 1, 2, \cdots, M-1). \tag{7}$$

For any fixed x and t such that $0 \leqq x \leqq 1$ and $t \geqq 0$, and for a given positive integer M, we can determine integers p and q such that

$$|x - p\Delta x| = \left|x - \frac{p}{M}\right| \leqq \frac{1}{2M}, \qquad |t - q\Delta t| = \left|t - \frac{qr}{M^2}\right| \leqq \frac{r}{2M^2}. \tag{8}$$

Let the coordinates of the corresponding mesh point $(p\Delta x, q\Delta t)$ be denoted by $x^{(M)}$ and $t^{(M)}$. The solution (6) takes on at this point the value

$$v^{(M)}(x^{(M)}, t^{(M)}) = \sum_{n=1}^{M-1} b_n^{(M)} \left(1 - 4r \sin^2 \frac{n\pi}{2M}\right)^{M^2 t^{(M)}/r} \sin n\pi x^{(M)}. \tag{9}$$

We shall say, for brevity, that a function $f(x)$ is *of class C* if $f(x)$ is continuous in (0, 1) except for a finite number of finite jumps, and is of bounded variation* in (0, 1).

The principal result to be established is then the following one:

THEOREM. *Let $f(x)$ be of class C, and suppose that r is fixed such that $0 < r < \frac{1}{2}$. Then, for any value of x in (0, 1), and for any positive value of t,*

$$\lim_{M\to\infty} v^{(M)}(x^{(M)}, t^{(M)}) = u(x, t). \tag{10}$$

It should be noticed that $v^{(M)}(x^{(M)}, t^{(M)})$ is the value of the solution of (2) at the mesh point nearest the point (x, t).

3. A preliminary result. We first establish the following proposition:

LEMMA. *Let $f(x)$ be of class C, and let a_n and $b_n^{(M)}$ be defined by equations (5) and (7), respectively. Then, corresponding to any $\epsilon > 0$, there exists an integer $M_0(\epsilon)$, independent of n, such that*

$$|a_n - b_n^{(M)}| < \epsilon \qquad (M > M_0) \tag{11}$$

when $n = 1, 2, \cdots, M-1$.

* A function is of bounded variation in (0, 1), in particular, if it possesses only a finite number of finite maxima and minima and finite jumps.

PROOF. Suppose that $f(x)$ possesses jumps J_α at the K points x_α interior to $(0, 1)$, such that $f(x_\alpha+) - f(x_\alpha-) = J_\alpha$ $(\alpha = 1, 2, \cdots, K)$, and is continuous elsewhere in $(0, 1)$, and define a function $\phi(x)$ by the equation

$$f(x) = A + Bx - \sum_{\alpha=1}^{K} J_\alpha s(x, x_\alpha) + \phi(x), \tag{12}$$

where $s(x, x_\alpha)$ is the step function

$$s(x, x_\alpha) = \begin{cases} 1, & x \leqq x_\alpha, \\ 0, & x > x_\alpha, \end{cases} \tag{13}$$

and where A and B are constants defined by the equations

$$A = f(0) + \sum_{\alpha=1}^{K} J_\alpha, \qquad B = f(1) - f(0) - \sum_{\alpha=1}^{K} J_\alpha. \tag{14}$$

It then follows immediately that $\phi(x)$ is continuous and of bounded variation in $(0, 1)$, and that $\phi(0) = \phi(1) = 0$.

Elementary calculations then lead to the results

$$a_n = (2/n\pi)[A(1 - \cos n\pi) - B \cos n\pi - \sum_{\alpha=1}^{K} J_\alpha(1 - \cos n\pi x_\alpha)] + \bar{a}_n \tag{15}$$

and

$$b_n^{(M)} = \frac{1}{M} \cot \frac{n\pi}{2M} \left[A(1 - \cos n\pi) - B \cos n\pi - \sum_{\alpha=1}^{K} J_\alpha \left(1 - \cos \frac{n\pi k_\alpha}{M}\right)\right] - \frac{1}{M} \sum_{\alpha=1}^{K} J_\alpha \sin \frac{n\pi k_\alpha}{M} + \bar{b}_n^{(M)}, \tag{16}$$

where k_α is the integral part of Mx_α, and where $\bar{a}_n$ and $\bar{b}_n^{(M)}$ are the coefficients relevant to $\phi(x)$,

$$\bar{a}_n = 2 \int_0^1 \phi(x) \sin n\pi x \, dx, \qquad \bar{b}_n^{(M)} = \frac{2}{M} \sum_{k=1}^{M-1} \phi\left(\frac{k}{M}\right) \sin \frac{n\pi k}{M}. \tag{17}$$

From (15) and (16) there follows, when $n < M$,

$$\begin{aligned} |a_n - b_n^{(M)}| \leqq{}& \frac{2}{n\pi} \left(1 - \frac{n\pi}{2M} \cot \frac{n\pi}{2M}\right) \\ &\cdot \left| A - (A + B) \cos n\pi - \sum_{\alpha=1}^{K} J_\alpha \left(1 - \cos \frac{n\pi k_\alpha}{M}\right)\right| \\ &+ \frac{2}{n\pi} \left| \sum_{\alpha=1}^{K} J_\alpha \left(\cos \frac{n\pi k_\alpha}{M} - \cos n\pi x_\alpha\right)\right| \\ &+ \frac{1}{M} \left| \sum_{\alpha=1}^{K} J_\alpha \sin \frac{n\pi k_\alpha}{M}\right| + |\bar{a}_n - \bar{b}_n^{(M)}|. \end{aligned} \tag{18}$$

If use is made of the fact that $(2/n\pi)[1 - (n\pi/2M) \cot (n\pi/2M)]$ is a positive increasing function of n for fixed M, when $0 < n < M$, and of the fact that k_α/M differs from x_α by not more than $1/2M$, so that

$$0 \leqq \frac{2}{n\pi} \left(1 - \frac{n\pi}{2M} \cot \frac{n\pi}{2M}\right) \leqq \frac{2}{M\pi}, \quad \left|\cos \frac{n\pi k_\alpha}{M} - \cos n\pi x_\alpha\right| \leqq 2 \sin \frac{n\pi}{4M} \leqq \frac{n\pi}{2M},$$

when $0 \leqq n \leqq M$, it may be deduced from (18) that

$$|a_n - b_n^{(M)}| \leqq \frac{2C}{M\pi} + \frac{J}{M} + \frac{J}{M} + |\bar{a}_n - \bar{b}_n^{(M)}|, \tag{19}$$

where J and C are constants, independent of n and M, defined by the equations

$$J = \sum_{\alpha=1}^{K} |J_\alpha|, \qquad C = 2|A| + |B| + 2J. \tag{20}$$

Let a positive number ϵ be assigned. It will be shown next that an integer M_0' can be determined, independently of n, such that

$$|\bar{a}_n - \bar{b}_n^{(M)}| < \frac{\epsilon}{2} \qquad (M > M_0'). \tag{21}$$

Temporarily assuming this fact, and defining M_0'' as any fixed integer larger than $(2C + 4J)/\epsilon$, we may deduce from (19) and (21) that

$$|a_n - b_n^{(M)}| < \epsilon \qquad (M > M_0), \tag{22}$$

where $M_0 = \max(M_0', M_0'')$, for $n = 1, 2, \cdots, M - 1$, and the lemma is established.

In order to establish (21), we notice the validity of the equation

$$\phi\left(\frac{\mu}{M}\right) = \sum_{n=1}^{\infty} \bar{a}_n \sin\frac{n\pi\mu}{M} = \sum_{n=1}^{M-1} \bar{b}_n^{(M)} \sin\frac{n\pi\mu}{M} \qquad (\mu = 1, 2, \cdots, M-1),$$

and the consequent validity of the equation

$$\sum_{n=1}^{M-1} (\bar{a}_n - \bar{b}_n^{(M)}) \sin\frac{n\pi\mu}{M} = -R_M\left(\frac{\mu}{M}\right) \qquad (\mu = 1, 2, \cdots, M-1), \tag{23}$$

where

$$R_M(x) \equiv \sum_{n=M}^{\infty} \bar{a}_n \sin n\pi x. \tag{24}$$

In virtue of the facts that $\phi(x)$ is continuous and of bounded variation in (0, 1), and that $\phi(0) = \phi(1) = 0$, the expansion

$$\phi(x) = \sum_{n=1}^{\infty} \bar{a}_n \sin n\pi x$$

converges *uniformly* in (0, 1). Hence, corresponding to any given $\epsilon > 0$, there exists an integer M_0' such that

$$|R_M(x)| < \tfrac{1}{4}\epsilon \qquad (M > M_0'), \tag{25}$$

where M_0' is independent of x when $0 \leqq x \leqq 1$. Hence there follows, from (23) and (25),

$$|\bar{a}_n - \bar{b}_n^{(M)}| = (2/M)\left|\sum_{k=1}^{M-1} R_M(k/M) \sin(n\pi k/M)\right| < \tfrac{1}{2}\epsilon \quad (M > M_0'), \tag{26}$$

as was to be shown.

4. Proof of the convergence theorem. We begin with the identity

$$\begin{aligned} v^{(M)}(x^{(M)}, t^{(M)}) - u(x, t) &= [u(x^{(M)}, t^{(M)}) - u(x, t)] \\ &+ [v^{(M)}(x^{(M)}, t^{(M)}) - u(x^{(M)}, t^{(M)})] \equiv \Delta_1 + \Delta_2. \end{aligned} \tag{27}$$

Because of the fact that $|x - x^{(M)}| \leq \frac{1}{2}M$ and $|t - t^{(M)}| \leq r/2M^2$, the continuity of $u(x, t)$ ensures the possibility of choosing an integer $M_1(\epsilon)$ such that

$$|\Delta_1| < \epsilon \qquad (M > M_1), \tag{28}$$

for fixed values of x and t, and arbitrary $\epsilon > 0$. In fact, the *uniformity* of the continuity for $0 \leq x \leq 1$ and $t \geq t_0 > 0$ guarantees that M_1 can be chosen independently of x and t when $t \geq t_0$.

In order to somewhat simplify the notation in the remainder of the proof, we introduce the abbreviations

$$\begin{gathered} A_n \equiv e^{-n^2\pi^2 t^{(M)}}, \qquad B_n \equiv [1 - 4r \sin^2 (n\pi/2M)]^{M^2 t^{(M)}/r}, \\ C_n \equiv B_n - A_n, \qquad S_n \equiv \sin n\pi x^{(M)}, \qquad b_n \equiv b_n^{(M)}, \end{gathered} \tag{29}$$

in accordance with which there follows

$$\Delta_2 = \sum_{n=1}^{M-1} b_n B_n S_n - \sum_{n=1}^{\infty} a_n A_n S_n . \tag{30}$$

It is convenient to rewrite this equation as follows:

$$\begin{aligned} \Delta_2 = {} & \sum_{n=1}^{M-1} (b_n - a_n) A_n S_n + \sum_{n=1}^{[M/3]} (b_n - a_n) C_n S_n + \sum_{n=[M/3]+1}^{M-1} (b_n - a_n) C_n S_n \\ & + \sum_{n=1}^{[M/3]} a_n C_n S_n + \sum_{n=[M/3]+1}^{M-1} a_n B_n S_n - \sum_{n=[M/3]+1}^{\infty} a_n A_n S_n \\ \equiv {} & \Delta_{21} + \Delta_{22} + \cdots + \Delta_{28}, \end{aligned} \tag{31}$$

where $[M/3]$ represents the integral part of $M/3$.

In consequence of the preceding lemma, there follows first

$$|\Delta_{21}| \leq \epsilon \sum_{n=1}^{M-1} e^{-n^2\pi^2 t} \leq \epsilon \sum_{n=1}^{M-1} e^{-n\pi^2 t} < \epsilon (1 - e^{-\pi^2 t_0})^{-1} \quad (M > M_1), \tag{32}$$

for all $t \geq t_0 > 0$ and $0 \leq x \leq 1$.

A lemma established by Leutert [3] guarantees that, for any given $\epsilon > 0$, there exists $M_2(\epsilon)$ such that

$$|(1 - 4r \sin^2 z)^{M^2 t/r} - e^{-4M^2 t z^2}| < \epsilon/M \qquad (M > M_2) \tag{33}$$

for any r such that $0 < r \leq \frac{1}{2}$, where M_2 is independent of z and t when $0 \leq z \leq \pi/6$ and $t \geq t_0 > 0$. The left-hand member of (33) is identified with $|C_n|$ by writing $z = n\pi/2M$ and replacing t by $t^{(M)}$. From this result and inequality (11), there follows

$$|\Delta_{22}| < (M/3) \cdot \epsilon \cdot (\epsilon/M) < \epsilon^2 \qquad (M > M_3), \tag{34}$$

where $M_3 = \max (M_0, M_2)$.

When $\pi/6 \leq z \leq \pi/2$ and $0 < r < \frac{1}{2}$, we have

$$|1 - 4r \sin^2 z| \leq \max (1 - r, 4r - 1) \equiv \rho < 1. \tag{35}$$

If also $t \geq t_0 > 0$, we have

$$e^{-4M^2 t z^2} \leq e^{-M^2 \pi^2 t_0/9}. \tag{36}$$

By writing $z = n\pi/2M$, it then follows that

$$|C_n| \leqq \rho^{M^2 t_0/r} + e^{-M^2 \pi^2 t_0/l} \tag{37}$$

when $M/3 \leqq n \leqq M$. It is clearly possible to choose $M_4(\epsilon)$ in such a way that each term in the right-hand member of (37) is smaller than ϵ/M when $M > M_4$. By using this result, together with (11), we may deduce that

$$|\Delta_{23}| < M \cdot \epsilon \cdot (2\epsilon/M) = 2\epsilon^2 \qquad (M > M_5), \tag{38}$$

where $M_5 = \max(M_0, M_4)$.

If use is made of the inequality

$$|a_n| \leqq 2\int_0^1 |f(x)|\,dx \leqq 2P, \tag{39}$$

where P is the maximum value of $|f(x)|$ in $(0, 1)$, it is found that

$$|\Delta_{24}| < \tfrac{1}{2}M \cdot 2P \cdot (\epsilon/M) < P\epsilon \qquad (M > M_2) \tag{40}$$

and

$$|\Delta_{25}| < M \cdot 2P \cdot (\epsilon/M) = 2P\epsilon \qquad (M > M_4). \tag{41}$$

Finally, from the uniform convergence of the series solution (4), when $t \geqq t_0 > 0$, it follows that $M_6(\epsilon)$ exists such that

$$|\Delta_{26}| < \epsilon \qquad (M > M_6), \tag{42}$$

where M_6 is independent of x and t for $0 \leqq x \leqq 1$ and $t \geqq t_0 > 0$.

Thus, for any $t \geqq t_0 > 0$, and for any x in $(0, 1)$, the preceding results guarantee that

$$|v^{(M)}(x^{(M)}, t^{(M)}) - u(x, t)| < \epsilon[1 + (1 - e^{-\pi^2 t_0})^{-1} + \epsilon + 2\epsilon + P + 2P + 1] \qquad (M > M_7), \tag{43}$$

where $M_7 = \max(M_0, M_1, \cdots, M_6)$, and the desired theorem follows from the arbitrariness of ϵ.

In fact, it is seen from the preceding developments that each M_i can be chosen independently of x and t, when $t \geqq t_0 > 0$. The convergence is thus *uniform* in x and t when $t \geqq t_0 > 0$.

5. Additional remarks. If $f(x)$ is such that the series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x \tag{44}$$

is *absolutely* convergent* for $0 \leqq x \leqq 1$, convergence of the solution of (2) to the solution of (1) can be readily established when $0 < r \leqq \frac{1}{2}$, the value $r = \frac{1}{2}$ now being included. In this case, the expression for $v^{(M)}$ can be expressed in the alternative form

* Sufficient conditions are that $f(0) = f(1) = 0$, that $f(x)$ be of bounded variation in $(0, 1)$, and that $f(x)$ satisfy a Lipschitz condition of positive order in $(0, 1)$.

$$v^{(M)}(x^{(M)}, t^{(M)}) = \sum_{n=1}^{\infty} a_n \left(1 - 4r \sin^2 \frac{n\pi}{2M}\right)^{M^2 t^{(M)}/r} \sin n\pi x^{(M)}, \tag{45}$$

convergence following from the fact that $| 1 - 4r \sin^2 (n\pi/2M) | \leqq 1$ for all n, when $0 < r \leqq \frac{1}{2}$, and we may write

$$v^{(M)}(x^{(M)}, t^{(M)}) - u(x, t) = \Delta_1 + \Delta_{21} + \Delta_{22} + \Delta_3, \tag{46}$$

where

$$\Delta_3 = \sum_{n=\lfloor M/2 \rfloor + 1}^{\infty} a_n \left(1 - 4r \sin^2 \frac{n\pi}{2M}\right)^{M^2 t^{(M)}/r} \sin n\pi x^{(M)}. \tag{47}$$

That $| \Delta_3 |$ can be made small by taking M large follows from the absolute convergence of (44), when $0 < r \leqq \frac{1}{2}$.

It is presumed that convergence when $r = \frac{1}{2}$ can be established by other methods under considerably less restrictive conditions on $f(x)$.

In conclusion, it is pointed out that the condition $0 < r \leqq \frac{1}{2}$ is not *necessary* for convergence of the solution of (2) to that of (1) as $M \to \infty$. That is, instability does not necessarily imply absence of strict convergence. In particular, if $f(x)$ is such that the series (44) terminates after a finite number of terms, the proof of convergence of $v^{(M)}(x^{(M)}, t^{(M)})$ to $u(x, t)$ for *any* fixed positive value of r is trivial. However, when $r > \frac{1}{2}$, convergence as M is increased generally cannot be realized in actual *numerical* calculation in any case, when the formulation (2) is used without modification, unless special precautions are taken to regulate the propagation of round-off errors.

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AN ERROR BOUND FOR THE NUMERICAL QUADRATURE OF ANALYTIC FUNCTIONS

By David Young[1]

In a previous paper[2] a five point formula

$$I_1(f) = \tfrac{1}{15}h\,\{24f(z_0) + 4[f(z_0 + h) + f(z_0 - h)] - [f(z_0 + ih) + f(z_0 - ih)]\} \tag{1}$$

was derived as a means for obtaining approximate numerical values for

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz \tag{2}$$

where $f(z)$ is analytic or harmonic. The five point formula was shown to yield exact results for polynomials of degree five or less. In this paper an upper bound for $|\,I(f) - I_1(f)\,|$, for $f(z)$ analytic will be derived.

We first prove

Theorem 1:[3] If $f(z)$ is analytic in (D) the domain of convexity[4] of $z_0, z_1, \cdots, z_n$, and if $P(z)$ is the polynomial of degree m such that

$$P^{(i)}(z_k) = f^{(i)}(z_k) \qquad (i = 0, 1, \cdots, \nu_k - 1) \tag{3}$$

where

$$\textstyle\sum_{k=0}^{n} \nu_k = m + 1$$

then

$$|\,f(z) - P(z)\,| \leq \frac{\prod_{k=0}^{n} |\,s - s_k\,|^{\nu_k}}{(m+1)!}\,Z_{m+1}; \qquad (z \in (D))$$

where Z_{m+1} is a point in the domain of convexity (S_{m+1}) corresponding to the domain covered by $f^{(m+1)}(t)$ when t assumes all values in (D).

Proof: The existence and uniqueness of $P(z)$ are well known.[5] By the Hermite formula[6] we have

$$f(z) - P(z) = \frac{1}{2\pi i}\int_C \frac{\omega(z)f(t)\,dt}{\omega(t)(t - z)} \tag{4}$$

[1] This work was done under Navy Contracts N5ori-76, Proj. 22, and N5ori-07634.

[2] Garrett Birkhoff and David Young, "Numerical Quadrature of Analytic and Harmonic Functions," J. Math. and Phys., **29**, 217–221, (1950).

[3] For the case of distinct points of interpolation this result is given in A. Lowan and H. Salzer, "Coefficients for Interpolation within a Square Grid in the Complex Plane," J. Math. and Phys., **23**, 156–166, (1944).

[4] The domain of convexity of a set of points is the intersection of all simply connected convex domains containing the given set.

[5] J. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," American Math. Society Colloq. Publications, Vol. 20, New York, 1935.

[6] See for example; Walsh, loc. cit., page 50.

where C is any contour in (D) which contains $z_0, z_1, \cdots, z_n, z$ in its interior and where

$$\omega(z) = \prod_{k=0}^{n} (z - z_k)^{\nu_k}$$

On the other hand for the case of $(m + 1)$ distinct points, Montel[7] has proved that if $P_1(z)$ is the polynomial which interpolates $f(z)$ at $(m + 1)$ distinct points, then

$$f(z) - P_1(z) = \prod_{i=0}^{m} (z - z_i) \frac{D_{m+2}(f)}{d_{m+2}} \tag{5}$$

where

$$D_{m+2}(f) = \begin{vmatrix} 1 & z & z^2 & \cdots & z^m & f(z) \\ 1 & z_0 & z_0^2 & \cdots & z_0^m & f(z_0) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & z_m & z_m^2 & \cdots & z_m^m & f(z_m) \end{vmatrix} \tag{6}$$

and

$$d_{m+2} = D_{m+2}(z^{m+1})$$

Montel[8] has also shown that

$$D_{m+2}(f)/d_{m+2} = Z_{m+1}/(m + 1)! \tag{7}$$

where Z_{m+1} is defined above. If the z_k are distinct, we have

$$f(z) - P_1(z) = \prod_{i=0}^{n} (z - z_i) Z_{m+1}/(m + 1)! \tag{8}$$

Now let $\{\zeta_0, \zeta_1, \cdots, \zeta_m\}$ be a sequence of points in (D) which includes z_k, ν_k times $(k = 0, 1, \cdots, n)$. By the continuity of the integrand of (4) it follows that

$$\lim_{z_i \to \zeta_i} [f(z) - P_1(z)] = [f(z) - P(z)] = \lim_{z_i \to \zeta_i} \frac{Z_{m+1}}{(m + 1)!} \omega(z) = \frac{Z_{m+1}^*}{(m + 1)!} \omega(z)$$

where Z_{m+1}^* belongs to (S_{m+1}). This proves Theorem 1.

Since $| Z_{m+1}^* | \leq \operatorname{Max}_{\xi \in (D)} | f^{(m+1)}(\xi) |$ we have

COROLLARY

$$| f(z) - P(z) | \leq \frac{\prod_{k=0}^{n} | z - z_k |^{\nu_k}}{(m + 1)!} \operatorname{Max}_{\xi \in (D)} | f^{(m+1)}(\xi) |$$

THEOREM 2

$$| I(f) - I_1(f) | \leq \tfrac{1}{1890} | h |^7 \operatorname{Max}_{z \in S} | (f^{(\nu)}(z)) |$$

where S denotes the square with vertices $z_0 + i^k h$ $(k = 0, 1, 2, 3)$.

[7] P. Montel, "Sur une Formule de Darboux et les Polynomes d'Interpolation," Annali della R. Scuola Normale Superiore di Pisa, serie II. vol. I, (1932).

[8] P. Montel, "Sur Quelques Propriétés des Différences Divisées," Journal de Mathématique Pures et Appliquées, 9 Serie, 16. 219–231, (1937).

It was shown in an earlier paper[9] that

$$I(P) = I_1(P) = I_1(f).$$

Therefore

$$|I(f) - I_1(f)| = |I(f) - I(P)| \leq \int_{x_0-h}^{x_0+h} |f(z) - P(z)|\,|dz|$$

$$\leq \int_{x_0-h}^{x_0+h} |(z-x_0)^2((z-x_0)^4 - h^4)|\,|dz| \cdot \operatorname*{Max}_{x \in S} |f^{(\nu+1)}(z)|/6!$$

The integral equals

$$\int_{-h}^{h} |t|^2 |t^4 - h^4|\,|dt| = \int_{|h|}^{|h|} s^2(|h|^4 - s^4)\,ds = \tfrac{8}{315}|h^7|$$

$$|I(f) - I(P)| \leq \operatorname*{Max}_{x \in S} \frac{f^{\nu+1}(z)}{6!} \cdot \tfrac{8}{315}|h|^7 = \operatorname*{Max}_{x \in S} \frac{f^{\nu+1}(z)}{1890}|h|^7$$

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[9] Birkhoff and Young, loc. cit.

A VARIATIONAL THEOREM FOR $\nabla^2 u + \lambda u = 0$ AND ITS APPLICATION*

By E. T. Kornhauser and I. Stakgold

1. Introduction. This paper is concerned with the study of certain properties of the eigenfunctions and eigenvalues of the partial differential equation $\nabla^2 u + \lambda u = 0$, subject to various boundary conditions.

The proofs and illustrations will be given for the two-dimensional case although they can often be extended to the case of an arbitrary number of independent variables.

Results will often be stated in terms of a corresponding physical problem: the propagation of electromagnetic waves in cylindrical waveguides of arbitrary cross section. In this case it is well known that Maxwell's equations reduce to the two-dimensional form:

$$\nabla^2 u + \lambda u = 0 \quad \text{in } R, \qquad u]_B = 0 \text{ for } E\text{-modes}$$

$$\nabla^2 v + \mu v = 0 \quad \text{in } R, \qquad \partial v/\partial n]_B = 0 \text{ for } H\text{-modes}$$

R is the region enclosed by the guide; B is the boundary wall of the guide; λ, μ are constant parameters, u, v, are functions from which the electromagnetic field is calculated.

The problem for E-modes will be called the *first* boundary-value problem; the problem for H-modes, the *second* boundary-value problem. The mathematical literature on these two problems is quite extensive [1]–[5]. A particularly complete treatment of known properties will be found in Courant-Hilbert [1].

The spectrum of the eigenvalues of the first problem forms a discrete set with the properties:

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$$

The corresponding eigenfunctions u_0, u_1, u_2, $\cdots$ form a complete orthonormal set over R.

For the second problem we again have a discrete spectrum:

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \cdots$$

The corresponding eigenfunctions v_0, v_1, v_2, $\cdots$ form a complete orthonormal set over R. We also note that $v_0 = \text{constant} = 1/\sqrt{A}$, where A is the area of R.

A well-known theorem (see ref. 1, p. 356) states: $\mu_k \leq \lambda_k$.

One of the aims of this paper is to show that

$$\mu_1 < \lambda_0$$

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This theorem admits of a simple physical interpretation. In waveguide terminology, each u_k defines an E-mode whose cut-off wavelength is given by $2\pi/\sqrt{\lambda_k}$; each v_k (except v_0, which represents the trivial case of vanishing electromagnetic field) defines an H-mode whose cutoff wavelength is given by $2\pi/\sqrt{\mu_k}$.

The largest possible value of $2\pi/\sqrt{\lambda_k}$ and $2\pi/\sqrt{\mu_k}$ defines the "absolute" cutoff wavelength. The corresponding mode is called the "dominant" mode. It is clear that the "absolute" cutoff wavelength is furnished by $2\pi/\sqrt{\lambda_0}$ or $2\pi/\sqrt{\mu_1}$, whichever is larger. If we can prove $\mu_1 < \lambda_0$, we will have proved a theorem often conjectured:

"For a cylindrical guide of arbitrary cross section, the dominant mode is always an H-mode."

In order to establish this theorem we will proceed according to the following plan:

(a) We will prove first that among all domains R of a *given* area A, μ_1 for the circle is the *largest*. Our demonstration uses, in part, a perturbation method, and therefore is not a rigorous mathematical proof.

(b) We then quote the theorem [6]–[10] that among all domains R of a *given* area A, λ_0 for the circle is the *smallest*.

(c) We then use the well-known result (obtained from the exact solutions) that $\mu_1 < \lambda_0$ for the circle.

It is clear that (a), (b), (c), taken together, prove that $\mu_1 < \lambda_0$ for any region R. Actually, much more has been proved. (a) by itself is a theorem which is of considerable mathematical interest.

2. Variational principle. As was previously mentioned the eigenvalues of the second boundary value problem form an enumerable sequence

$$0 = \mu_0 < \mu_1 \leq \mu_2 \cdots$$

and $v_0 = 1/\sqrt{A}$. It is possible for the eigenvalue μ_1 to be degenerate (e.g., circle, square, etc.). If an eigenvalue is degenerate, it is listed as often as its degree of degeneracy (i.e., as often as the number of linearly independent eigenfunctions corresponding to the particular eigenvalue). The degree of degeneracy is finite [11].

The eigenvalues and eigenfunctions of the second boundary value problem are characterized by the following variational principle: Among all functions $\phi(x, y) \in C_1$ (i.e., continuous and with sectionally continuous partial derivatives), which satisfy the conditions

$$\int_R \phi v_0 \, dx \, dy = 0, \cdots, \int_R \phi v_{k-1} \, dx \, dy = 0,$$

the function which minimizes the functional

$$H(\phi) = \frac{\int_R \left| \text{grad } \phi \right|^2 dx \, dy}{\int_R \phi^2 \, dx \, dy}$$

is $v_k(x, y)$; in addition,

$$\mu_k = \text{minimal value of } H = H(v_k).$$

Remark: Note that *no* boundary conditions are imposed on the functions ϕ which are admitted for comparison. The minimum automatically occurs for a function which has

$$\frac{\partial \phi}{\partial n}\Big]_B = 0.$$

This is the so-called free, or natural, boundary condition. (The name stems from the obvious physical significance of the boundary condition for the problem of the membrane governed by $\nabla^2 u + \lambda u = 0$.) In mathematical language, it is an *unstable* boundary condition [12], since it drops out in the variational formulation.

In the case of the first boundary value problem it is known that among all regions R of a given area, the circle yields the lowest value of λ_0 [6]–[10]. Column 1 of the accompanying table, which indicates the relative values of λ_0 for various regions of the same area, suggests the validity of this theorem which was first conjectured by Rayleigh [6]. Column 2, indicating the relative values of μ_1 for regions of the same area, suggests the theorem which is to be proved in section III.

Finally, it is clear that if $\bar{R}$ and R are similar regions, then:

$$\frac{\bar{\lambda}_k}{\lambda_k} = \frac{\bar{\mu}_k}{\mu_k} = \frac{A}{\bar{A}}.$$

TABLE 1

	$\frac{\lambda}{\lambda_{\text{circle}}}$	$\frac{\mu}{\mu_{\text{circle}}}$
Circle	1	1
Square	1.09	.93
Quadrant of circle	1.14	.70
Confocal parabolas	1.18	.57
Equilateral triangle	1.25	.72
Semicircle	1.26	.51
2:1 rectangle (or isosceles right triangle)	1.36	.47

3. The maximizing theorem and its application. The theorem to be proved is the following:

Among all plane regions of area A, the circle yields the largest value of μ_1.

The proof of the theorem is divided in two parts:

1. Infinitesimal, area-preserving deformations of the circle never increase the value of μ_1.

2. Given any region not a circle, there exists an infinitesimal, area-preserving transformation which increases the value of μ_1.

It is clear that 1 and 2 together prove our main theorem if we are willing to make the following plausible assumption:

"Among all plane regions of area A, there exists some region which yields the largest possible value of μ_1."

We prove part 1 of our main theorem:

Let R be a circle of radius a. We denote its boundary by B, and its lowest non-zero eigenvalue by μ_1. Let R' be the region obtained by an infinitesimal area-preserving deformation of the boundary B. The corresponding eigenvalue of R' is μ_1'. In polar coordinates, the equation of the new boundary B' is expressed by

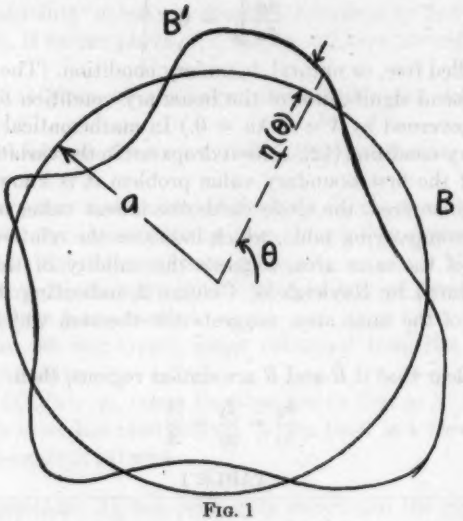

FIG. 1

$r = a + \eta(\theta)$, where $\eta(\theta)$ is small compared to a (see Fig. 1). Since the areas of R and R' are equal, we must have

$$\int_0^{2\pi} \eta(\theta)\, d\theta = 0.$$

We will now construct a function $\psi(r, \theta)$ such that $\int_{R'} \psi\, dA = 0$, and with the property:

$$\frac{\int_{R'} |\operatorname{grad} \psi|^2\, dA}{\int_{R'} \psi^2\, dA} \leq \mu_1.$$

By the variational principle, we know:

$$\frac{\int_{R'} |\operatorname{grad} \psi|^2\, dA}{\int_{R'} \psi^2\, dA} \geq \mu_1'.$$

Hence we will have proved $\mu_1' \leq \mu_1$, which is part 1 of our theorem.

$\psi(r, \theta)$ is constructed as follows: Let $\phi(r, \theta)$ be any v_1 eigenfunction for the circle R. It is well known that $\phi(r, \theta) = F(r) \cos(\theta + \alpha)$, where $F(r)$ is a known function of r and α is an arbitrary real parameter representing the degeneracy of v_1 for the circle. The function $\phi(r, \theta)$ is seen to be analytic for all values of r and θ. Consider $\phi(r, \theta)$ on R'. This is not an acceptable comparison function for R' since $\int_{R'} \phi \, dA$ does not necessarily vanish.

Let $\psi(r, \theta) = \phi(r, \theta) + \epsilon$, where ϵ is a constant to be chosen such that

$$\int_{R'} \psi \, dA = 0.$$

We have:

$$\int_{R'} (\phi + \epsilon) \, dA = \pi a^2 \epsilon + \int_{R'} \phi \, dA.$$

If we denote the values of $\phi(r, \theta)$ on the boundary B of R by $M \cos(\theta + \alpha)$; then

$$\int_{R'} \phi \, dA \cong \int_0^{2\pi} M \cos(\theta + \alpha) a\eta(\theta) \, d\theta.$$

Therefore:

$$\epsilon = -\frac{M}{\pi a} \int_0^{2\pi} \eta(\theta) \cos(\theta + \alpha) \, d\theta$$

An upper bound for μ_1' is

$$\int_{R'} |\operatorname{grad} \psi|^2 \, dA \Big/ \int_{R'} \psi^2 \, dA; \quad \text{and} \quad \mu_1 = \int_R |\operatorname{grad} \phi|^2 \, dA \Big/ \int_R \phi^2 \, dA.$$

Therefore,

$$\mu_1' - \mu_1 \leq \frac{\int_{R'} |\operatorname{grad} \psi|^2 \, dA}{\int_{R'} \psi^2 \, dA} - \frac{\int_R |\operatorname{grad} \phi|^2 \, dA}{\int_R \phi^2 \, dA}$$

$$= \frac{\int_{R'} |\operatorname{grad} \phi|^2 \, dA + \Delta \int |\operatorname{grad} \phi|^2 \, dA}{\int_R \phi^2 \, dA + \Delta \int \phi^2} - \frac{\int_R |\operatorname{grad} \phi|^2 \, dA}{\int_R \phi^2 \, dA}.$$

The Δ's represent first-order changes. When we expand the leading fraction in a power series, we obtain (keeping first-order terms only):

$$\mu_1' - \mu_1 \leq \Delta \int |\operatorname{grad} \phi|^2 \, dA - \mu_1 \Delta \int \phi^2 \, dA.$$

We note that $|\operatorname{grad} \phi| = |\operatorname{grad} \psi|$ in the region common to R and R'. In the remaining region, we note that $\partial\psi/\partial r = \partial\phi/\partial r \cong 0$ since $\partial\phi/\partial n]_B = 0$.

We conclude:

$$\Delta \int |\operatorname{grad} \phi|^2 = \int_0^{2\pi} (1/a^2)(\partial\phi/\partial\theta)^2 \eta(\theta) a \, d\theta = (M^2/a) \int_0^{2\pi} \eta(\theta) \sin^2(\theta + \alpha) \, d\theta.$$

Since $\psi^2 = \phi^2 + 2\epsilon\phi + \epsilon^2$, we have:

$$\Delta \int \phi^2 = \int_0^{2\pi} M^2 \cos^2(\theta + \alpha) a\eta(\theta) \, d\theta + 2\epsilon \int_0^{2\pi} M \cos(\theta + \alpha) a\eta(\theta) \, d\theta + \pi a^2 \epsilon^2$$

$$= M^2 a \int_0^{2\pi} \cos^2(\theta + \alpha) \eta(\theta) \, d\theta - \pi a^2 \epsilon^2.$$

Hence, to the first order,

$$\Delta \int \phi^2 = M^2 a \int_0^{2\pi} \cos^2(\theta + \alpha) \eta(\theta) \, d\theta.$$

This implies

$$(a/M^2)(\mu_1' - \mu_1) \leq \int_0^{2\pi} \eta \sin^2(\theta + \alpha) \, d\theta - \mu_1 a^2 \int_0^{2\pi} \eta \cos^2(\theta + \alpha) \, d\theta.$$

This inequality holds for every value of α. We will show that we can choose α such that the right side is negative, thereby proving part 1. We have the relationship

$$0 = \int_0^{2\pi} \eta(\theta) \, d\theta = \int_0^{2\pi} \eta(\theta)[\sin^2(\theta + \alpha) + \cos^2(\theta + \alpha)] \, d\theta$$

Hence

$$(a/M^2)(\mu_1' - \mu_1) \leq (1 + \mu_1 a^2) \int_0^{2\pi} \eta \sin^2(\theta + \alpha) \, d\theta$$

$$= -(1 + \mu_1 a^2) \int_0^{2\pi} \eta \cos^2(\theta + \alpha) \, d\theta$$

Two cases arise:

(i) $\int_0^{2\pi} \eta \sin^2(\theta + \alpha) \, d\theta = 0$ for each α, and $\mu_1' \leq \mu_1$

(ii) for some α_0, $\int_0^{2\pi} \eta \sin^2(\theta + \alpha_0) \, d\theta \neq 0$, say > 0.

Consider $\alpha_1 = \alpha_0 + \frac{1}{2}\pi$, $\sin(\theta + \alpha_1) = \sin(\theta + \frac{1}{2}\pi + \alpha_0) = -\cos(\theta + \alpha_0)$.

$$\int_0^{2\pi} \eta \sin^2(\theta + \alpha_1) \, d\theta = \int_0^{2\pi} \eta \cos^2(\theta + \alpha_0) \, d\theta = -\int_0^{2\pi} \eta \sin^2(\theta + \alpha_0) \, d\theta;$$

hence

$$(a/M^2)(\mu_1' - \mu_1) \leq -(1 + \mu_1 a^2) \int_0^{2\pi} \eta \sin^2(\theta + \alpha_0) \, d\theta < 0 \qquad (\text{since } \mu_1 > 0);$$

hence $\mu_1' < \mu_1$.

We have thus proved part 1 of our theorem. We note that case (i) will certainly arise when $\eta(\theta) = \delta \sin(\theta + \theta_0)$, i.e., when the deformation preserves the circular shape; this is exactly the result to be expected.

We will now prove part 2 of the theorem.

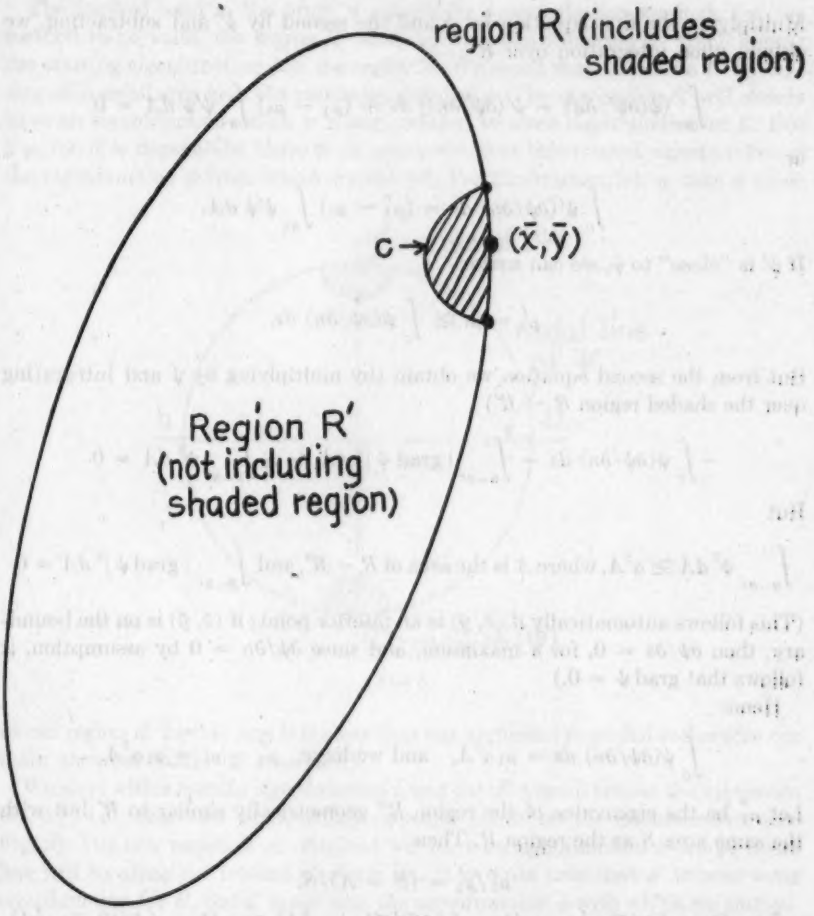

FIG. 2

Given a region R which is not a circle, there exists an infinitesimal area-preserving deformation which increases μ_1. Let $\psi(x, y)$ be the v_1 eigenfunction corresponding to R. Let the maximum of $|\psi(x, y)|$ over R occur at a point $(\bar{x}, \bar{y})$ the maximal value being denoted by α. We construct R' by cutting off an arbitrarily shaped small area about $(\bar{x}, \bar{y})$ (see Fig. 2). The area of R' is less than that of R. Later we will expand the region R', preserving its shape, so that its area will equal that of R.

Let $\psi'(x, y)$ be the ν_1 eigenfunction corresponding to R';

$$\nabla^2\psi' + \mu_1'\psi' = 0 \qquad \partial\psi'/\partial n]_{B'} = 0$$

$$\nabla^2\psi + \mu_1\psi = 0 \qquad \partial\psi/\partial n]_B = 0.$$

Multiplying the first equation by ψ and the second by ψ' and subtracting, we obtain, upon integration over R':

$$\int_{B'} (\psi(\partial\psi'/\partial n) - \psi'(\partial\psi/\partial n))\, ds + (\mu_1' - \mu_1) \int_{R'} \psi'\psi\, dA = 0$$

or

$$\int_C \psi'(\partial\psi/\partial n)\, ds = (\mu_1' - \mu_1) \int_{R'} \psi'\psi\, dA.$$

If ψ' is "close" to ψ, we can write:

$$\mu_1' - \mu_1 \cong \int_C \psi(\partial\psi/\partial n)\, ds.$$

But from the second equation we obtain (by multiplying by ψ and integrating over the shaded region $R - R'$)

$$-\int_C \psi(\partial\psi/\partial n)\, ds - \int_{R-R'} |\operatorname{grad}\psi|^2\, dA + \mu_1 \int_{R-R'} \psi^2\, dA = 0.$$

But

$\int_{R-R'} \psi^2\, dA \cong \alpha^2 A$, where A is the area of $R - R'$, and $\int_{R-R'} |\operatorname{grad}\psi|^2\, dA = 0.$

(This follows automatically if $(\bar{x}, \bar{y})$ is an interior point; if $(\bar{x}, \bar{y})$ is on the boundary, then $\partial\psi/\partial s = 0$, for a maximum, and since $\partial\psi/\partial n = 0$ by assumption, it follows that grad $\psi = 0$.)

Hence

$$\int_C \psi(\partial\psi/\partial n)\, ds = \mu_1\alpha^2 A, \quad \text{and we have} \quad \mu_1' - \mu_1 = \mu_1\alpha^2 A.$$

Let μ_1'' be the eigenvalue of the region R'' geometrically similar to R' but with the same area S as the region R. Then

$$\mu_1''/\mu_1' = (S - A)/S.$$

$$\mu_1'' = [1 - (A/S)\, \mu_1' = \mu_1[1 - (A/S)]\, (1 + \alpha^2 A) \cong \mu_1[1 - (A/S) + \alpha^2 A]$$

$$\mu_1'' - \mu_1 = \mu_1[\alpha^2 A - (A/S)]$$

We have $\int_R \psi^2\, dA = 1$, and $|\psi| \leq \alpha$; since ψ is not constant, these imply: $\alpha^2 S > 1$, or $\alpha^2 > 1/S$.

Therefore $\mu_1'' - \mu_1 > 0$.

We have thus exhibited an infinitesimal area-preserving deformation which increases μ_1. This proves part 2 of our theorem.

Note: The above proof is valid only in the case of regions R for which the eigenvalue μ_1 is nondegenerate (which is the usual case).

The method used in the proof is essentially a perturbation method. For the method to be valid, the region R' must have an eigenfunction ψ' which is near the starting eigenfunction ψ of the region R. We recall that we obtain R' by cutting off a small area near the maximum point of ψ. The new region R' will clearly have an eigenfunction which is closely related to some eigenfunction of R. But if μ_1 for R is degenerate, there is no guarantee that this related eigenfunction is the eigenfunction ψ from which we cut off. For illustration, let us take a circle

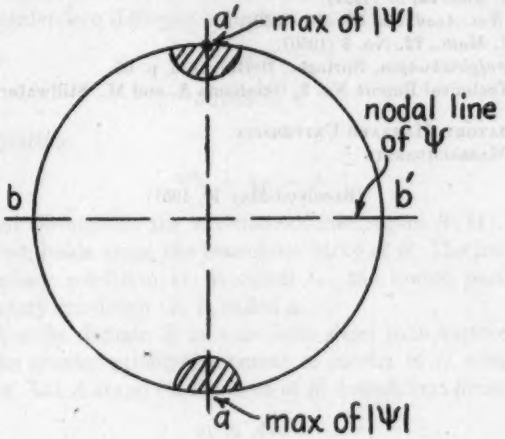

FIG. 3

as our region R. In this case it is clear that our argument must fail (otherwise our main theorem would be incorrect!).

We start with a specific eigenfunction ψ and cut off a small area at the maximum of $|\psi|$, i.e., near two diametrically opposite points on the circumference (see Fig. 3). The new region R' so obtained will have an eigenfunction ψ' whose nodal line will be along aa' instead of along bb'. It is quite true that ψ' is near some eigenfunction for R, but ψ' is *not* near the eigenfunction ψ with which we started.

We have thus proved that among all regions of given area A the circle yields the largest value of μ_1.*

It has been proved [6]–[10] that among all regions of a given area A the circle yields the least value of λ_0.

* A similar calculation has been carried out for the three-dimensional form of the second boundary-value problem, with the result that the sphere yields the largest value of μ_1 for regions of equal volume.

As is easily seen from an examination of the correct solutions for the circle, $\lambda_0 \cong 5.78/a^2$, $\mu_1 \cong 3.38/a^2$; therefore, for the circle, $\mu_1 < \lambda_0$. Hence for any other region R, it follows that $\mu_1 < \lambda_0$.

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REMARKS ON THE FOREGOING PAPER[1]

By G. Pólya

Abstract. The wave equation in two dimensions is considered under two different boundary conditions: (1) the solution vanishes on the boundary, (2) the normal derivative of the solution vanishes on the boundary. It is fully proved that the first positive eigenvalue under condition (2) is lower than the first eigenvalue under condition (1). The proof substitutes for an argument of Kornhauser and Stakgold a simpler and completely rigorous argument which yields, by the way, also a practical estimate.

1. Adopting the notation of the foregoing paper by E. T. Kornhauser and I. Stakgold[2], I consider two different boundary conditions

$$u = 0, \tag{1}$$

$$\partial u/\partial n = 0 \tag{2}$$

for the wave equation

$$\nabla^2 u + \lambda u = 0 \tag{3}$$

which is satisfied throughout the two-dimensional region R; (1), or (2), whichever is considered, holds along the boundary curve of R. The lowest eigenvalue under the boundary condition (1) is called λ_0, the lowest *positive* eigenvalue under the boundary condition (2) is called μ_1.

Let us conceive the domain R as a uniform sheet with surface-density 1 and let I' denote the greater principal moment of inertia of R with respect to its center of gravity. Let A stand for the area of R. I shall first prove that

$$\mu_1 \leqq A/I'. \tag{4}$$

(This is a crude, but easily obtainable and quite practical, estimate.) Then I shall proceed in three steps:

(a) Of all domains R with a given area A the circle has the minimum I'.

(b) Of all domains R with a given area A the circle has the minimum λ_0.

(c) For the circle $A/I' < \lambda_0$.

From these facts, it follows immediately that

$$\mu_1 < \lambda_0 \tag{5}$$

for *any* domain. This scheme is very similar to that of Kornhauser and Stakgold (see the Introduction of the foregoing paper). The highly interesting statement concerning the maximum of μ_1 for given A discovered, but not rigorously proved, by Kornhauser and Stakgold plays, however, no role in my proof for (5).

By the way, the argument just sketched extends to three or more dimensions.

[1] The present paper was written under the sponsorship of the Office of Naval Research.

[2] It was known to me in its original form as Technical Report No. 117, written under O.N.R. contract N5ori-75, T.O. No. 1.

2. Let ϕ denote a function defined in R with continuous partial derivatives ϕ_x and ϕ_y that satisfies the condition

$$\iint \phi dxdy = 0; \tag{6}$$

the double integral in (6), as all the other double integrals in the present section 2, is extended over the region R. Then

$$\mu_1 \leqq \frac{\iint (\phi_x^2 + \phi_y^2)\, dxdy}{\iint \phi^2\, dxdy}. \tag{7}$$

Let us assume that the center of gravity of R is at the origin, that the coordinate axes are the principal axes of inertia at this point, and that the moment of inertia about the y-axis is not less than that about the x-axis. Then

$$\iint x dxdy = 0; \tag{8}$$

$$I' = \iint x^2 dxdy \geqq \iint y^2 dxdy. \tag{9}$$

In view of (8) the simple function $\phi = x$ satisfies the condition (6) and so the predicted inequality (4) follows at once from (7) and (9). It is easy to see that the case of equality cannot be attained in (4), but we do not need this fact here.

Let us observe that (9) yields

$$2I' \geqq \iint (x^2 + y^2)\, dxdy = I \tag{10}$$

where I is the polar moment of inertia of R with respect to its center of gravity.

3. Let R' denote the circle with center at the origin and area A. The polar moment of inertia of R' with respect to its center of gravity is $A^2/2\pi$. Therefore

$$\begin{aligned} I - A^2/2\pi &= \left(\iint_R - \iint_{R'}\right)(x^2 + y^2)\, dxdy \\ &= \left(\iint_E - \iint_{E'}\right)(x^2 + y^2)\, dxdy; \end{aligned} \tag{11}$$

E stands for that part of the region R which is exterior to the circle R', and E' for that part of R' which does not belong to R. The second line of (11) follows from the first because, obviously, the common part of the regions R and R' does not contribute to the difference displayed in the first line. The area of R is equal to that of R' and, therefore, the area of E is equal to that of E'. Yet the integrand $x^2 + y^2$ is greater in any interior point of E than in any interior point of E' (since E' is within and E without the circle). Therefore, by (11),

$$I > A^2/2\pi \tag{12}$$

unless R coincides with R'.[3] From (10) and (12) follows that

$$I' \geqq A^2/4\pi. \tag{13}$$

4. The point (b) stated in the introduction was conjectured by Lord Rayleigh and first proved by Faber and Krahn;[4] it can be expressed by the inequality

$$\lambda_0 \geqq (2.4048)^2\pi/A \tag{14}$$

where 2.4048 is the first root of the Bessel function $J_0(x)$. The plan involving the points (a), (b), (c) sketched in the introduction is now translated into the following chain of inequalities, in which we use successively (14), (13), and (4):

$$\lambda_0 \geqq (2.4048)^2\pi/A = (1.2024)^2 4\pi/A \geqq (1.2024)^2 A/I' \geqq (1.2024)^2\mu_1 .$$

This proves amply the desired (5).

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[3] A sharper result is given in G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton, 1951, pp. 152–153.

[4] For sharper results and further literature see G. Pólya and G. Szegö, l.c., especially pp. 191–193.

TABLES OF VALUES OF THE INTEGRALS

$$\int_0^\infty \frac{x^m}{\sinh^p x}\,dx \text{ and } \int_0^\infty \frac{x^m}{\cosh^p x}\,dx$$

By Chih-Bing Ling

In dealing with plane problems in elasticity, it is often found necessary to evaluate certain integrals which can be developed into series in terms of the integral $\int_0^\infty \frac{x^m}{\sinh^p x}\,dx$. In this note, values of this integral together with those of a similar integral in which the function $\sinh x$ is replaced by $\cosh x$ will be tabulated.

For convenience, denote

$$\begin{aligned} I(m, p) &= \frac{p^{m+1}}{2^p(m!)} \int_0^\infty \frac{x^m\,dx}{\sinh^p x}, \qquad (m \geq p \geq 1) \\ J(m, p) &= \frac{p^{m+1}}{2^p(m!)} \int_0^\infty \frac{x^m\,dx}{\cosh^p x}, \qquad (m \geq 0, p \geq 1) \end{aligned} \tag{1}$$

where m and p are integers under restrictions as indicated. Both functions tend to unity as m tends to infinity.

The following recurrence formulas are readily found by integration by parts:

$$\begin{aligned} I(m, p) &= \frac{p}{4(p-1)}\left(\frac{p}{p-2}\right)^m \{I(m-2, p-2) - I(m, p-2)\} \\ J(m, p) &= -\frac{p}{4(p-1)}\left(\frac{p}{p-2}\right)^m \{J(m-2, p-2) - J(m, p-2)\} \end{aligned} \tag{2}$$

The two leading functions with respect to p are found equal to

$$\begin{aligned} I(m, 1) &= 1 + \frac{1}{3^{m+1}} + \frac{1}{5^{m+1}} + \frac{1}{7^{m+1}} + \cdots = U_{m+1}, \qquad (m \geq 1) \\ I(m, 2) &= 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \cdots = S_m, \qquad (m \geq 2) \\ J(m, 1) &= 1 - \frac{1}{3^{m+1}} + \frac{1}{5^{m+1}} - \frac{1}{7^{m+1}} + \cdots = u_{m+1}, \qquad (m \geq 0) \\ J(m, 2) &= 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \cdots = s_m, \qquad (m \geq 1). \end{aligned} \tag{3}$$

S_m is commonly known as Riemann Zeta function. By successive substitution, we find for $p \geq 1$,

$$
\begin{aligned}
I(m, 2p+1) &= \frac{(2p+1)^{m+1}}{2^{2p}(2p)!} \sum_{r=0}^{p} (-1)^r a_r U_{m-2p+2r+1}, && (m \geq 2p+1) \\
I(m, 2p+2) &= \frac{(p+1)^{m+1}}{(2p+1)!} \sum_{r=0}^{p} (-1)^r b_r S_{m-2p+2r}, && (m \geq 2p+2) \\
J(m, 2p+1) &= \frac{(2p+1)^{m+1}}{2^{2p}(2p)!} \sum_{r=0}^{p} (-1)^{p+r} a_r u_{m-2p+2r+1}, && (m \geq 2p) \\
J(m, 2p+2) &= \frac{(p+1)^{m+1}}{(2p+1)!} \sum_{r=0}^{p} (-1)^{p+r} b_r s_{m-2p+2r}, && (m \geq 2p+1)
\end{aligned} \tag{4}
$$

where a_r and b_r are the coefficients of x^r given by

$$
\begin{aligned}
(1+x)(1+3^2x)(1+5^2x)\cdots\{1+(2p-1)^2x\} &= \sum_{r=0}^{p} a_r x^r \\
(1+x)(1+2^2x)(1+3^2x)\cdots(1+p^2x) &= \sum_{r=0}^{p} b_r x^r
\end{aligned} \tag{5}
$$

When $x = -1$, the left hand sides of the expressions vanish identically. Thus,

$$
\sum_{r=0}^{p} (-1)^r a_r = 0, \qquad \sum_{r=0}^{p} (-1)^r b_r = 0 \tag{6}
$$

Therefore, if we denote

$$
\begin{aligned}
U'_m &= U_m - 1, & u'_m &= 1 - u_m \\
S'_m &= S_m - 1, & s'_m &= 1 - s_m
\end{aligned} \tag{7}
$$

Equations (4) become, for $p \geq 1$,

$$
\begin{aligned}
I(m, 2p+1) &= \frac{(2p+1)^{m+1}}{2^{2p}(2p)!} \sum_{r=0}^{p} (-1)^r a_r U'_{m-2p+2r+1}, && (m \geq 2p+1) \\
I(m, 2p+2) &= \frac{(p+1)^{m+1}}{(2p+1)!} \sum_{r=0}^{p} (-1)^r b_r S'_{m-2p+2r}, && (m \geq 2p+2) \\
J(m, 2p+1) &= \frac{(2p+1)^{m+1}}{2^{2p}(2p)!} \sum_{r=0}^{p} (-1)^{p+r+1} a_r u'_{m-2p+2r+1}, && (m \geq 2p) \\
J(m, 2p+2) &= \frac{(p+1)^{m+1}}{(2p+1)!} \sum_{r=0}^{p} (-1)^{p+r+1} b_r s'_{m-2p+2r+1}, && (m \geq 2p+1)
\end{aligned} \tag{8}
$$

The values of U'_m, S'_m and s'_m were tabulated by Glaisher to 32 decimal places (ref. 1), and u'_m by the same author to 18 decimal places (ref. 2). With these tables, values of the two integrals are evaluated and shown in Tables I and II.

However, the preceding evaluation does not cover those values of $J(m, p)$ in which $p > m + 1$. In evaluating them, two leading functions with respect to m are required. We find for $p \geq 1$,

TABLE I

Values of $I(m, p)$

m	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
1	1.23370	—	—	—	—	—	—	—
2	1.05180	1.64493	—	—	—	—	—	—
3	1.01468	1.20206	2.21760	—	—	—	—	—
4	1.00452	1.08232	1.43601	3.00059	—	—	—	—
5	1.00145	1.03693	1.20567	1.76138	4.06673	—	—	—
6	1.00047	1.01734	1.10777	1.38624	2.19851	5.51653	—	—
7	1.00016	1.00835	1.05952	1.21935	1.63155	2.77832	7.48711	—
8	1.00005	1.00408	1.03386	1.13201	1.37524	1.95474	3.54366	10.16505
9	1.00002	1.00201	1.01961	1.08218	1.23745	1.58199	2.37486	4.55231
10	1.00001	1.00099	1.01149	1.05226	1.15602	1.37965	1.84950	2.91780
11	1.00000	1.00049	1.00678	1.03370	1.10501	1.25804	1.56431	2.19146
12	1.00000	1.00025	1.00402	1.02194	1.07184	1.18015	1.39190	1.79943
13	1.00000	1.00012	1.00240	1.01438	1.04971	1.12811	1.28028	1.56262
14	1.00000	1.00006	1.00143	1.00946	1.03497	1.09230	1.20458	1.40879
15	1.00000	1.00003	1.00085	1.00625	1.02433	1.06713	1.15154	1.30374

TABLE II

Values of $J(m, p)$

m	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
0	0.78540	0.50000	0.29452	0.16667	0.092039	0.050000	0.026845	0.014286
1	0.91597	0.69315	0.46796	0.29543	0.17863	0.10482	0.060178	0.033979
2	0.96895	0.82245	0.61947	0.42996	0.28207	0.17772	0.10864	0.064874
3	0.98894	0.90154	0.73891	0.55572	0.39200	0.26354	0.17077	0.10749
4	0.99616	0.94703	0.82655	0.66435	0.49933	0.35599	0.24343	0.16097
5	0.99869	0.97212	0.88762	0.75282	0.59762	0.44902	0.32255	0.22332
6	0.99955	0.98555	0.92857	0.82172	0.68328	0.53777	0.40397	0.29182
7	0.99985	0.99259	0.95526	0.87356	0.75505	0.61887	0.48408	0.36356
8	0.99995	0.99623	0.97228	0.91152	0.81333	0.69044	0.55980	0.43572
9	0.99998	0.99809	0.98297	0.93875	0.85948	0.75182	0.62930	0.50590
10	0.99999	0.99904	0.98960	0.95796	0.89528	0.80323	0.69141	0.57225
11	1.00000	0.99952	0.99368	0.97133	0.92261	0.84546	0.74569	0.63346
12	1.00000	0.99976	0.99617	0.98055	0.94319	0.87959	0.79225	0.68876
13	1.00000	0.99988	0.99769	0.98686	0.95853	0.90682	0.83157	0.73786
14	1.00000	0.99994	0.99861	0.99115	0.96986	0.92829	0.86434	0.78079
15	1.00000	0.99997	0.99916	0.99405	0.97818	0.94507	0.89134	0.81783

$$J(0, p) = \frac{\pi^{\frac{1}{2}} p\, \Gamma(\frac{1}{2}p)}{2^{p+2}\Gamma(\frac{1}{2}p + \frac{1}{2})}$$
$$J(1, p+2) = \frac{(p+2)^2}{4p(p+1)}\left\{J(1, p) - \frac{1}{2^p}\right\} \tag{9}$$

The second relation indicates that these functions can be expressed in terms of $J(1, 1)$ or $J(1, 2)$ as follows: For $p \geq 1$,

TABLE III

The factor $2^p(m!)/p^{m+1}$

m	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
0	2.00000	2.00000	2.66667	4.00000	6.40000	1.06667	1.82857×10	3.20000×10
1	2.00000	1.00000	8.88889×10^{-1}	1.00000	1.28000	1.77778	2.61224	4.00000
2	4.00000	1.00000	5.92593×10^{-1}	5.00000×10^{-1}	5.12000×10^{-1}	5.92593×10^{-1}	7.46356×10^{-1}	1.00000
3	1.20000×10	1.50000	5.92593×10^{-1}	3.75000×10^{-1}	3.07200×10^{-1}	2.96296×10^{-1}	3.19867×10^{-1}	3.75000×10^{-1}
4	4.80000×10	3.00000	7.90123×10^{-1}	3.75000×10^{-1}	2.45760×10^{-1}	1.97531×10^{-1}	1.82781×10^{-1}	1.87500×10^{-1}
5	2.40000×10^{2}	7.50000	1.31687	4.68750×10^{-1}	2.45760×10^{-1}	1.64609×10^{-1}	1.30558×10^{-1}	1.17188×10^{-1}
6	1.44000×10^{3}	2.25000×10	2.63374	7.03125×10^{-1}	2.94912×10^{-1}	1.64609×10^{-1}	1.11907×10^{-1}	8.78906×10^{-2}
7	1.00800×10^{4}	7.87500×10	6.14540	1.23047	4.12877×10^{-1}	1.92044×10^{-1}	1.11907×10^{-1}	7.69043×10^{-2}
8	8.06400×10^{4}	3.15000×10^{2}	1.63877×10	2.46094	6.60603×10^{-1}	2.56059×10^{-1}	1.27893×10^{-1}	7.69043×10^{-2}
9	7.25760×10^{5}	1.41750×10^{3}	4.91632×10	5.53711	1.18909	3.84088×10^{-1}	1.64434×10^{-1}	8.65173×10^{-2}
10	7.25760×10^{6}	7.08750×10^{3}	1.63877×10^{2}	1.38428×10	2.37817	6.40146×10^{-1}	2.34906×10^{-1}	1.08147×10^{-1}
11	7.98336×10^{7}	3.89813×10^{4}	6.00884×10^{2}	3.80676×10	5.23197	1.17360	3.69138×10^{-1}	1.48702×10^{-1}
12	9.58003×10^{8}	2.33888×10^{5}	2.40354×10^{3}	1.14203×10^{2}	1.25567×10	2.34720	6.32809×10^{-1}	2.23053×10^{-1}
13	1.24540×10^{10}	1.52027×10^{6}	1.04153×10^{4}	3.71159×10^{2}	3.26475×10	5.08561	1.17522	3.62460×10^{-1}
14	1.74357×10^{11}	1.06419×10^{7}	4.86048×10^{4}	1.29906×10^{3}	9.14131×10	1.18664×10	2.35043	6.34306×10^{-1}
15	2.61535×10^{12}	7.98141×10^{7}	2.43024×10^{5}	4.87147×10^{3}	2.74239×10^{2}	2.96660×10	5.03664	1.18932

$$J(1, 2p+1) = \frac{(2p+1)\Gamma(p+\frac{3}{2})}{2^{2p+1}(p!)}\left\{\frac{4u_2}{\pi^{\frac{1}{2}}} - \sum_{r=1}^{p}\frac{(r-1)!}{(2r-1)\Gamma(r+\frac{1}{2})}\right\}$$
$$J(1, 2p+2) = \frac{(2p+2)(p+1)!}{2^{2p+2}\,\Gamma(p+\frac{3}{2})}\left\{\pi^{\frac{1}{2}}\log 2 - \sum_{r=1}^{p}\frac{\Gamma(r+\frac{1}{2})}{2r(r!)}\right\} \tag{10}$$

By the second recurrence formula in Equations (2), the rest of $J(m, p)$ are evaluated. Table III shows the factor $2^p(m!)/p^{m+1}$.

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ON THE UNIQUENESS OF GAS FLOWS*

By J. L. Ericksen

1. Introduction. We seek conditions that two plane, stationary, irrotational, ideal gas flows have the same streamline patterns. We will show that, in general, the velocity fields of the two flows must be strictly proportional and, moreover, the adiabatic exponents for the two flows must be identical. Exceptions to this may occur only if the flows are of vortex type, the streamlines being either parallel straight lines or concentric circles, or if the flows are of source-sink type. This extends to the case of compressible fluids results of Gilbarg[1] and Prim[2,3], who have treated the corresponding problem for the stationary flow of incompressible ideal fluids and related problems concerning the streamline patterns common to two different classes of flows.

It is known[4] that the only flows of the type we are considering which are non-isentropic are of the vortex type. With this in mind, we consider only isentropic flows.

Where not otherwise specified, rectangular coordinates are used, and the notation of Cartesian tensors is employed. The equations governing the motion are then[5]

$$u_{\alpha,\beta} = u_{\beta,\alpha} \tag{1}$$

$$u_{\alpha,\beta}u_\alpha u_\beta = c^2 u_{\sigma,\sigma}\,, \tag{2}$$

$$c^2 = \gamma p/\rho = (\gamma - 1)(a^2 - v^2)/2, \tag{3}$$

$$p = N\rho^\gamma, \tag{4}$$

where u_α is the velocity vector, ρ the density, p the pressure, c the sound speed, and $v = \sqrt{u_\alpha u_\alpha}$. The quantities a^2, N, and γ are constants. For convenience we assume $\gamma > 1$ in what follows, though it is sufficient to require merely $\gamma \neq 1$. From (3) and (4) one easily finds that

$$\rho = [(\gamma - 1)/2\gamma N]^{1/(\gamma-1)}(a^2 - v^2)^{1/(\gamma-1)}. \tag{5}$$

We may introduce a velocity potential, ϕ, and a stream function, ψ, satisfying

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the relations $\phi_{,\alpha} = u_\alpha$, $\psi_{,1} = -\rho u_2$, $\psi_{,2} = \rho u_1$. With ϕ and ψ as coordinates, equations (1) and (2) become respectively

$$\theta_\phi = \rho v_\psi / v, \tag{6}$$

$$\theta_\psi = (v^2/c^2 - 1)v_\phi/\rho v, \tag{7}$$

where $\theta = \tan^{-1}(u_2/u_1)$ is the angle made by the velocity vector with the positive x_1-axis. Differentiating (6) with respect to ψ, (7) with respect to ϕ, and eliminating θ, one obtains, using (3) and (5),

$$\rho^2 c^2[c^2(vv_\phi)_\psi - v_\phi^2(v^2 + 2c^2)] = v_\phi^2(\gamma v^4 - c^2 v^2 + 2c^4) + c^2(v^2 - c^2)(vv_\phi)_\phi. \tag{8}$$

In the following, we assume that ρ and v are strictly positive, and that all functions appearing are continuous. By "flow" we mean a solution of equations (1) through (4).

2. Conditions for two flows to have the same streamlines. Consider any given flow, and a second flow with the same streamlines. Denoting by bars the quantities associated with the second flow, we must have $\bar\phi = f(\phi)$ and $\bar\psi = g(\psi)$. Noting that $\bar v^2 = \bar\phi_{,\alpha}\bar\phi_{,\alpha}$ and $\bar\rho^2\bar v^2 = \bar\psi_{,\alpha}\bar\psi_{,\alpha}$, one easily obtains

$$\bar v = Av, \tag{9}$$

$$\bar\rho\bar v = B\rho v, \tag{10}$$

$$\bar\rho = (B/A)\rho, \tag{11}$$

with $A = A(\phi) = |df/d\phi|$, $B = B(\psi) = |dg/d\psi|$. From (5), (9), and (11),

$$k(\bar a^2 - A^2 v^2)^{1/(\bar\gamma-1)} = (B/A)(a^2 - v^2)^{1/(\gamma-1)}, \tag{12}$$

where k is a positive constant. Differentiating (12) logarithmically with respect to ϕ, then ψ, and using (3), one finds that

$$(\bar c^2 - A^2 c^2)vv_\phi A = A' c^2(\bar v^2 - \bar c^2), \tag{13}$$

$$(\bar c^2 - A^2 c^2)vv_\psi B = B' c^2 \bar c^2, \tag{14}$$

where the primes denote differentiation. Now for any flow with the velocity potential ϕ, stream function ψ, and for arbitrary constants c_1, c_2, c_3 and c_4, we can always choose $\bar a$ and $\bar N$ so that, with $\bar\gamma = \gamma$, $\bar\phi = c_1\phi + c_2$, $\bar\psi = c_3\psi + c_4$ are, respectively, the velocity potential and stream function of a flow, provided only that $c_1 c_3 > 0$. This leads us to call a flow *unique* if, for every other flow with the same streamlines, $A' = B' = \bar\gamma - \gamma = 0$. We will then prove the following

THEOREM: *If a flow is not unique, then either $v = v(\phi)$ or $v = v(\psi)$.*

In a domain where $\bar c^2 = A^2 c^2$, one finds, using (3) and (9), that

$$(\bar\gamma - 1)\bar a^2 - (\gamma - 1)a^2 A^2 + (\gamma - \bar\gamma)A^2 v^2 = 0. \tag{15}$$

Remembering that A is a function of ϕ alone, we see that either v is a function of ϕ alone, or $\gamma = \bar{\gamma}$ and A is constant. Also, from (14), B is constant. Next consider a domain where A is constant, but $\bar{c}^2 \neq A^2c^2$. By (13), $v_\phi = 0$, so v is a function of ψ alone. In the remainder of this section we require that, in the domain under consideration, neither A' nor $\bar{c}^2 - A^2c^2$ vanish. Under these conditions we will show that we must have $v = v(\phi)$. We will omit some of the rather tedious, but elementary, calculations involved in the proof and record only the essential steps.

Since $\bar{c}^2 \neq A^2c^2$, (13) and (14) may be solved for vv_ϕ and vv_ψ. Differentiation of the resulting formulas yields, using (3), (9), (13), and (14)

$$(vv_\psi)_\psi = B''c^2\bar{c}^2/B(\bar{c}^2 - A^2c^2) - [(B')^2c^2\bar{c}^2/B^2(\bar{c}^2 - A^2c^2)^2] \cdot[\gamma\bar{c}^2 + (\bar{\gamma} - 2)A^2c^2 + (\gamma - \bar{\gamma})A^2c^2\bar{c}^2/(\bar{c}^2 - A^2c^2)], \tag{16}$$

$$\begin{aligned}(vv_\phi)_\phi = {} & A''c^2(\bar{v}^2 - \bar{c}^2)/A(\bar{c}^2 - A^2c^2) + [(A')^2c^2/A^2(\bar{c}^2 - A^2c^2)^2] \\ & \cdot[-(\gamma - 1)(\bar{v}^2 - \bar{c}^2)^2 + (\bar{\gamma} + 1)\bar{c}^2(v^2 - c^2)A^2 \\ & - (\bar{v}^2 - \bar{c}^2)(\bar{c}^2 - A^2c^2) - (\gamma - \bar{\gamma})(\bar{v}^2 - \bar{c}^2)A^2\bar{c}^2(v^2 - c^2)/(\bar{c}^2 - A^2c^2) \\ & + (\gamma - 1)a^2A^2(\bar{v}^2 - \bar{c}^2)].\end{aligned} \tag{17}$$

Substituting from (13), (14), (16), and (17) in (8) and simplifying, we have

$$\begin{aligned}B''/B = {} & [(B')^2/B^2(\bar{c}^2 - A^2c^2)][(\gamma + 1)\bar{c}^2 + (\bar{\gamma} - 2)A^2c^2 \\ & + (\gamma - \bar{\gamma})A^2c^2\bar{c}^2/(\bar{c}^2 - A^2c^2) + 2c^2\bar{c}^2/v^2] \\ & + (1/\rho^2c^2\bar{c}^2)\{(\bar{v}^2 - \bar{c}^2)(v^2 - c^2)A''/A + [(A')^2/A^2(\bar{c}^2 - A^2c^2)] \\ & \cdot[(\bar{v}^2 - \bar{c}^2)(v^2 + (\gamma - 2)c^2 + 2c^4/v^2) \\ & + (v^2 - c^2)(\bar{\gamma}\bar{c}^2\bar{v}^2 - (\bar{\gamma} + 2)\bar{c}^2A^2c^2 + \bar{v}^2A^2c^2 + \bar{c}^4) \\ & - (\gamma - \bar{\gamma})(\bar{v}^2 - \bar{c}^2)A^2c^2(v^2 - c^2)^2/(\bar{c}^2 - A^2c^2) \\ & + (\gamma - 1)a^2A^2(\bar{v}^2 - \bar{c}^2)(v^2 - c^2)]\}.\end{aligned} \tag{18}$$

By means of (3) and (9), (18) can be put in the form

$$\begin{aligned}B''/B = {} & [(B')^2/B^2(\bar{c}^2 - A^2c^2)^2v^2][\textstyle\sum_{i=0}^{2} F_i v^{2i} \\ & + (1/\rho^2c^2\bar{c}^2\bar{v}^2(\bar{c}^2 - A^2c^2)^2][\textstyle\sum_{i=4}^{6} F_i v^{2i-6}],\end{aligned} \tag{19}$$

where the functions F_i referred to here and in the following are functions of ϕ alone. Those essential to the proof are tabulated in section 4. The basic idea in the proof is to eliminate B, B', and B'' from (19) by differentiating with respect to ϕ, leading to a functional relation between v and certain functions of ϕ, and then show that this requires that $v = v(\phi)$. Differentiating (19) with respect to ϕ, using (3), (5), (9), and (13), one obtains

$$(B'/B)^2Y_1 + Y_2 = 0, \tag{20}$$

with

$$Y_1 = -2\bar{v}^2[(\gamma-\bar{\gamma})A^2\bar{c}^2(v^2-c^2)-(\gamma-1)A^2a^2(\bar{c}^2-A^2c^2)][\textstyle\sum_{i=0}^{8}F_iv^{2i}]$$
$$+A^2c^2(\bar{v}^2-\bar{c}^2)(\bar{c}^2-A^2c^2)[-2F_0+2F_2v^4+4F_3v^6$$
$$+(A/A')\bar{v}^2(\bar{c}^2-A^2c^2)^2[\textstyle\sum_{i=0}^{8}F_i'v^{2i}], \tag{21}$$

$$\rho^2c^2\bar{c}^2Y_2 = \{(\bar{c}^2-A^2c^2)v^2[(\gamma+\bar{\gamma})\bar{v}^2+(\gamma+1)\bar{c}^2-(\bar{\gamma}-1)A^2c^2]$$
$$-2\bar{c}^2(v^2-c^2)(\bar{c}^2-A^2c^2)-2\bar{v}^2[(\gamma-\bar{\gamma})\bar{c}^2(v^2-c^2)$$
$$-(\gamma-1)a^2(\bar{c}^2-A^2c^2)]\}[\textstyle\sum_{i=4}^{9}F_iv^{2i-8}]+(A/A')v^2(\bar{c}^2-A^2c^2)^2 \tag{22}$$
$$\cdot[\textstyle\sum_{i=4}^{9}F_i'v^{2i-8}]+[\sum_{i=5}^{9}(2i-8)F_iv^{2i-8}]c^2(\bar{c}^2-A^2c^2)(\bar{v}^2-\bar{c}^2).$$

Making use of (3) and (9), (21) and (22) may be written in the form

$$Y_1 = \textstyle\sum_{i=10}^{16}F_iv^{2i-20}, \tag{23}$$

$$\rho^2c^2\bar{c}^2Y_2 = \textstyle\sum_{i=17}^{16}F_iv^{2i-34}, \tag{24}$$

where, as mentioned previously, the F_i are functions of ϕ. Now, from (20)

$$Y_2(\partial Y_1/\partial\phi) = Y_1(\partial Y_2/\partial\phi). \tag{25}$$

Substituting in (25) from (23) and (24) and making use of (3), (5), (9), and (13), one finds that

$$\{[\textstyle\sum_{i=10}^{16}F_i'v^{2i-20}](A/A')(\bar{c}^2-A^2c^2)+[\sum_{i=11}^{16}(2i-20)F_iv^{2i-22}]c^2(\bar{v}^2$$
$$-\bar{c}^2)\}[\textstyle\sum_{i=17}^{25}F_iv^{2i-34}] = \{[(\gamma+\bar{\gamma})\bar{v}^2-(\gamma+1)\bar{c}^2-(\bar{\gamma}$$
$$-1)A^2c^2][\textstyle\sum_{i=17}^{25}F_iv^{2i-34}]+[\sum_{i=17}^{25}F_i'v^{2i-34}](A/A')(\bar{c}^2-A^2c^2) \tag{26}$$
$$+[\textstyle\sum_{i=18}^{25}(2i-34)F_iv^{2i-36}]c^2(\bar{v}^2-\bar{c}^2)\}[\sum_{i=10}^{16}F_iv^{2i-20}],$$

which, by means of (3) and (9) can be put in the more compact form

$$\textstyle\sum_{i=0}^{15}G_i(\phi)v^{2i} = 0, \tag{27}$$

the G_i being functions of $A, F_1, \cdots, F_{25}$ and derivatives of these functions. To prove our assertion that $v = v(\phi)$, it suffices to show that not all the G_i vanish identically. As is indicated in section 4,

$$G_0 = 2l^6l^5(\gamma+1)A^2(A')^2H, \tag{28}$$

where $2l = (\gamma-1)a^2$, $2l = (\bar{\gamma}-1)\bar{a}^2$, $H = 2(l-lA^2)$. Since A is, by assumption, not constant, G_0 can not vanish identically, thus $v = v(\phi)$. This completes the proof of our theorem.

3. Geometric Interpretation. If $v = v(\psi)$, the streamlines are either parallel straight lines or concentric circles.[6] We have left to consider the case where $v = v(\phi)$.

[6] Nemenyi, P., and Prim, R. C., *Some properties of plane rotational gas flows*, Journal of Math. and Physics, **27**, 1948, pp. 130–135.

Let $w_2 = -u_2$, $w_2 = u_1$. Let E denote an equipotential and its arc length, increasing in the direction given by w_α. Then the curvature, K, of E satisfies the relation

$$d(u_\alpha/v)/ds = (u_\alpha/v)_{,\beta}w_\beta/v = Kw_\alpha/v, \tag{29}$$

where K is positive (negative) if the center of curvature of E lies in the direction given by $-u_\alpha(u_\alpha)$. From (29), multiplying by w_α and summing, we obtain

$$u_{\alpha,\beta}w_\alpha w_\beta = v^3K. \tag{30}$$

Recalling that $w_1 = u_2$, $w_2 = u_1$, one easily verifies that

$$u_{\alpha,\beta}u_\alpha u_\beta - v^2u_{v,v} = -u_{\alpha,\beta}w_\alpha w_\beta = -v^3K. \tag{31}$$

Also

$$u_{\alpha,\beta}u_\alpha u_\beta = vv_{,\beta}u^\beta = v^3v_\phi. \tag{32}$$

Now if $v = v(\phi)$, v is constant on E, and by (3), so is c^2. Using (32), then (2), one sees that the left side of (31) is constant along E, from which it follows that K is constant along E. The equipotentials must thus be either circles or straight lines. From (6), $\theta = \theta(\psi)$, from which the streamlines are straight. The flow is thus of the source-sink type, the equipotentials being concentric circles, or the streamlines are parallel straight lines. Using these results and the theorem of the previous section we may state the

THEOREM: *A flow which is not unique must be either of the vortex type or the source-sink type.*

From the remark made in the introduction, there has been no loss of generality in assuming the flows isentropic.

4. The functions F_i and G_0. The functions F_i referred to in section 2 which are essential to the proof are those used in calculating G_0. Letting $2l = (\gamma - 1)a^2$, $2\bar{l} = (\bar{\gamma} - 1)\bar{a}^2$, $H = 2(\bar{l} - lA^2)$, these are

(1) $F_0 = llH$,

(2) $F_4 = l^2\bar{l}^2(A')^2H$,

(3) $F_{10} = l^2\bar{l}^2A^2H^2$

(4) $F_{11} = llHA^2\{-(\gamma - 1)\bar{l}^2 + 2l\bar{l}[1 + 2(\gamma - \bar{\gamma})]A^2 + l^2(\bar{\gamma} - 3)A^4\} - l^2\bar{l}A^4H^2$,

(5) $F_{17} = l^3\bar{l}^3(A')^2H^2$,

(6) $F_{18} = l^3\bar{l}^3(A')^2A^2H\,[\bar{l}(6 + 3\gamma - 5\bar{\gamma}) + A^2\bar{l}(2\bar{\gamma} - 6)] + l^2\bar{l}^2A\,[A''H - 2\bar{l}(A')^2A]\,H^2/2.$

It is also convenient to have the following derivatives:

$$F'_{10}A/A' = 2l^2\bar{l}^2A^2H\,[H - 4lA^2], \tag{7}$$

$$F'_{17}A/A' = 2l^3\bar{l}^3A\,[HA'' - 4\bar{l}(A')^2A]H. \tag{8}$$

G_0 can then be calculated from

$$G_0 = F_{17}[F'_{10}AH/2A' - 2llF_{11}] + F_{10}\{F_{17}(\gamma+1)a^2 + (\gamma-1)a^2A^2](\bar{\gamma}-1)/2 - F'_{17}AH/2A' + 2llF_{18}\}, \tag{9}$$

from which

$$G_0 = 2l^6\bar{l}^6(\gamma+1)A^2(A')^2H^4. \tag{10}$$

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ON THE VIBRATION AND ELASTIC STABILITY OF A RECTANGULAR PLATE CLAMPED AT ITS FOUR EDGES

By K. Munakata

1. Introduction and summary. In the present paper the writer presents a new method of solution suitable for discussing, in a unified manner, various problems concerning a rectangular plate clamped at its four edges, such as the fundamental and higher modes of its free vibration and the buckling under compressive or shearing forces. The inside of the rectangular plate is first transformed conformally into a circle, and then the problem is dealt with in the plane of the circle. The transformation naturally gives rise to various modifications in the differential equation as well as in the boundary conditions. If we make use of a complete set of orthogonal functions, the solution in the plane of the circle can be expanded into an infinite series which is convergent in the said circle.

An approximate solution, composed of a few of these orthogonal functions, has been determined by the aid of Galerkin's method, and then, by solving a determinantal equation, the value of the frequency of vibration has been determined as a function of the forces acting at the edges of the plate. The process of analysis has been greatly simplified by the use of complex variables.

Up to the present, detailed numerical computations have been made for the cases of free vibration and buckling as well as for some combined problems of these, confining ourselves to the modes of deformation symmetrical about the center of the plate. It has thus been found that our results for the case of a square plate are in fairly good agreement with the known accurate results obtained by Tomotika [1], Young [2]*, Weinstein [3] and Timoshenko [4].

In view of the fact, however, that a full description of our analysis, together with results obtained, needs a considerable space, we here confine ourselves to presenting the method of analysis by taking up the case of vibration of a rectangular plate only.

The present problem has been suggested by Prof. S. Tomotika, to whom the writer wishes to express his cordial thanks for his continual guidance and encouragement throughout the work. He is also indebted to Dr. K. Tamada for valuable discussions. The work was supported by a grant in aid for fundamental scientific research given by the Ministry of Education of Japan.

2. Preliminaries. In the first place the well-known solution for the free vibration of a thin homogeneous isotropic circular plate of radius unity with uniform thickness is given briefly for later use. When the plate vibrates in a normal mode the lateral displacement W assumes the form $W = w \cos(pt + \epsilon)$, and

* Prof. E. Reissner has kindly informed the writer of some recent papers on related subjects, together with the numerical results given in them. The writer wishes to express his sincere thanks to him.

w satisfies the equation:

$$\Delta\Delta w - k^4 w = 0, \tag{1}$$

where $\Delta = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial\theta^2)$, r and θ being the polar coordinates with the origin at the center of the plate. In the case of clamped edges the boundary conditions are

$$w = 0, \quad \partial w/\partial r = 0 \quad \text{at} \quad r = 1. \tag{2}$$

The particular solution of (1) can be expressed in terms of Bessel functions and is given by the real or imaginary part of the following expression:

$$w_{n\nu} = \{J_n(k_{n\nu}r) + \lambda_{n\nu} I_n(k_{n\nu}r)\} e^{in\theta}, \tag{3}$$

where $k_{n\nu}$ and $\lambda_{n\nu}$ are constants to be determined by the equations:

$$J_n(k_{n\nu})/I_n(k_{n\nu}) = J'_n(k_{n\nu})/I'_n(k_{n\nu}) = -\lambda_{n\nu}, \tag{4}$$

which follow immediately from the boundary conditions (2). $w_{n\nu}$ is called an eigenfunction corresponding to an eigenvalue $k^4_{n\nu}$. The suffixes n and ν, both assuming zero or positive integers, indicate the number of nodal diameters and nodal circles respectively.

The functions $w_{n\nu}$ as a whole constitute a complete system of orthogonal functions, and concerning these the following theorem is known to hold good [5]:

Any function which has continuous derivatives up to the fourth order on and inside a unit circle and satisfies the boundary conditions (2), can be expanded in terms of eigenfunctions (3) into an infinite series absolutely and uniformly convergent in the said circle.

Now, the interior of a rectangle with sides $2a$, $2b$ in a z-plane is mapped, conformally and one-to-one, into the interior of a unit circle in a ζ-plane by the relation:

$$z = A \int_0^\zeta \frac{d\zeta}{\sqrt{1 - 2\gamma\zeta^2 + \zeta^4}}, \tag{5}$$

where

$$A = \frac{2a}{(1+\kappa)K}, \qquad \gamma = 1 - 2\left(\frac{1-\kappa}{1+\kappa}\right)^2, \tag{6}$$

and K is the complete elliptic integral of the first kind with modulus κ [6].

3. Dynamical equations. Returning now to our proper problem, let us consider a thin homogeneous isotropic rectangular plate of sides $2a$, $2b$ with uniform thickness $2h$, and let the lateral displacement of a point in its middle plane be denoted by W. Then the dynamical equation for the free lateral vibration of the plate is given by

$$\partial^4 W/\partial x^4 + 2\partial^4 W/\partial x^2\partial y^2 + \partial^4 W/\partial y^4 + (2\rho h/D)\partial^2 W/\partial t^2 = 0, \tag{7}$$

where ρ and D are the density and the flexural rigidity of the plate respectively.

If the plate vibrates in a normal mode we can put $W = w \cos(pt + \epsilon)$ and the normal function w is then determined by the equation:

$$16(\partial^4 w/\partial z^2 \partial \bar{z}^2) - k_1^2 w = 0, \qquad k_1^2 = (2\rho h/D)p^2 \tag{8}$$

where we have introduced the complex variable $z = x + iy$ and its conjugate $\bar{z} = x - iy$ in place of x and y. Since we are considering the case of clamped edges in this paper the boundary conditions are given by

$$\begin{aligned} w &= 0, \quad \partial w/\partial x = 0 \quad \text{at} \quad x = \pm a; \\ w &= 0, \quad \partial w/\partial y = 0 \quad \text{at} \quad y = \pm b. \end{aligned} \tag{9}$$

Henceforth we consider the matter in the interior of the unit circle in the ζ-plane. From equation (5) we find $\partial/\partial z = A^{-1}\sqrt{1 - 2\gamma\zeta^2 + \zeta^4}\,\partial/\partial\zeta$, and applying this to equation (8), we obtain the fundamental equation in the ζ-plane in the form:

$$\begin{aligned} &16(1 - 2\gamma\zeta^2 + \zeta^4)(1 - 2\gamma\bar{\zeta}^2 + \bar{\zeta}^4)\frac{\partial^4 w}{\partial\zeta^2 \partial\bar{\zeta}^2} \\ &\quad - 32(1 - 2\gamma\zeta^2 + \zeta^4)(\gamma\bar{\zeta} - \bar{\zeta}^3)\frac{\partial^3 w}{\partial\zeta^2 \partial\bar{\zeta}} \\ &\quad - 32(1 - 2\gamma\bar{\zeta}^2 + \bar{\zeta}^4)(\gamma\zeta - \zeta^3)\frac{\partial^3 w}{\partial\zeta \partial\bar{\zeta}^2} \\ &\quad + 64(\gamma\zeta - \zeta^3)(\gamma\bar{\zeta} - \bar{\zeta}^3)\frac{\partial^2 w}{\partial\zeta \partial\bar{\zeta}} - k^2 w = 0, \end{aligned} \tag{10}$$

with

$$k^2 = (2\rho h A^4/D)p^2. \tag{11}$$

In view of the conformal correspondence between the z- and ζ-planes, it can easily be seen that the boundary conditions in the ζ-plane become

$$w = 0, \qquad \partial w/\partial r = 0 \qquad \text{at} \qquad r = 1. \tag{12}$$

4. Method of solution. Now, the theorem given in §2 can be applied to our problem, if we exclude the nearest neighborhood of four singular points which correspond to the vertices of the rectangle. Namely, we can expand the required solution in terms of eigenfunctions given by (3), which will be called 'coordinate-functions' in the following.

Thus, we put for an approximate solution

$$w = R \sum_{n=0}^{N} \sum_{\nu=0}^{M} A_{n\nu} w_{n\nu}, \tag{13}$$

where R means 'the real part of' and $A_{n\nu} = C_{n\nu} + iD_{n\nu}$ with real constants $C_{n\nu}$ and $D_{n\nu}$. Though there may be a few different methods for determining the coefficients of expansion, we shall here conveniently use Galerkin's method [7].

Thus, if we substitute the expression (13) in the left-hand side of the fundamental equation (10), it cannot become zero everywhere in the circle. Let the

residual be denoted by Ω and call it provisionally an 'error' in the equation. In order to make this error as small as possible, it is required that the error is orthogonal to each coordinate-function. Thus,

$$\left.\begin{matrix} R \\ I \end{matrix}\right\} \int_0^1 \int_0^{2\pi} \Omega w_{m\mu} r \, dr \, d\theta = 0, \qquad \begin{pmatrix} m = 0, 1, \cdots N, \\ \mu = 0, 1, \cdots M, \end{pmatrix} \tag{14}$$

which yield a system of linear homogeneous equations with unknown constants $C_{n\nu}$ and $D_{n\nu}$. The value of frequency can thus be obtained by solving a determinantal equation with the coefficients of these unknown constants as its elements.

In order to derive the above-mentioned error in the equation, it is necessary to calculate the derivatives of the coordinate-functions. They can be obtained easily as follows. Bearing the relation $\zeta = re^{i\theta}$ in mind, equation (3) can be expressed by the complex variables in the form:

$$w_{n\nu} = \{J_n(k_{n\nu}\sqrt{\zeta\bar{\zeta}}) + \lambda_{n\nu} I_n(k_{n\nu}\sqrt{\zeta\bar{\zeta}})\}(\zeta/\bar{\zeta})^{n/2}.$$

Differentiating this with respect to ζ or $\bar{\zeta}$ and making use of recurrence formulas for Bessel functions, the following expressions can be verified:

$$\begin{aligned} \partial w_{n\nu}/\partial\zeta &= \tfrac{1}{2} k_{n\nu}\{J_{n-1}(k_{n\nu}r) + \lambda_{n\nu} I_{n-1}(k_{n\nu}r)\} e^{i(n-1)\theta}, \\ \partial w_{n\nu}/\partial\bar{\zeta} &= -\tfrac{1}{2} k_{n\nu}\{J_{n+1}(k_{n\nu}r) - \lambda_{n\nu} I_{n+1}(k_{n\nu}r)\} e^{i(n+1)\theta}. \end{aligned} \tag{15}$$

If we put for brevity

$$\begin{aligned} \Phi_l(n\nu) &= J_l(k_{n\nu}r) + \lambda_{n\nu} I_l(k_{n\nu}r), \\ \Phi_l^*(n\nu) &= J_l(k_{n\nu}r) - \lambda_{n\nu} I_l(k_{n\nu}r), \end{aligned} \tag{16}$$

the coordinate-function becomes

$$w_{n\nu} = \Phi_n(n\nu) e^{in\theta}, \tag{17}$$

while equations (15) may be written as:

$$\begin{aligned} \partial w_{n\nu}/\partial\zeta &= \tfrac{1}{2} k_{n\nu} \Phi_{n-1}(n\nu) e^{i(n-1)\theta}, \\ \partial w_{n\nu}/\partial\bar{\zeta} &= -\tfrac{1}{2} k_{n\nu} \Phi_{n+1}^*(n\nu) e^{i(n+1)\theta}. \end{aligned} \tag{18}$$

The necessary successive derivatives can also be obtained immediately. We thus have

$$\begin{aligned} \partial^2 w_{n\nu}/\partial\zeta\partial\bar{\zeta} &= -\tfrac{1}{4} k_{n\nu}^2 \Phi_n^*(n\nu) e^{in\theta}, \\ \partial^3 w_{n\nu}/\partial\zeta^2\partial\bar{\zeta} &= -\tfrac{1}{8} k_{n\nu}^3 \Phi_{n-1}^*(n\nu) e^{i(n-1)\theta}, \\ \partial^3 w_{n\nu}/\partial\zeta\partial\bar{\zeta}^2 &= \tfrac{1}{8} k_{n\nu}^3 \Phi_{n+1}(n\nu) e^{i(n+1)\theta}, \\ \partial^4 w_{n\nu}/\partial\zeta^2\partial\bar{\zeta}^2 &= \tfrac{1}{16} k_{n\nu}^4 \Phi_n(n\nu) e^{in\theta}. \end{aligned} \tag{19}$$

Now, it is not difficult to write down the expression for Ω in the real form. After performing integration with respect to θ in equations (14), it can easily

be seen that the resulting expressions are composed of definite integrals of the form:

$$\int_0^1 r^k \Phi_l(n\nu)\Phi_m(m\mu)\,dr \qquad \text{and} \qquad \int_0^1 r^k \Phi_l^*(n\nu)\Phi_m(m\mu)\,dr$$

in the coefficients of $C_{n\nu}$ or $D_{n\nu}$. These integrals can be evaluated numerically by making use of appropriate tables of Bessel functions. Lastly, by solving a determinantal equation the values of k^2 can be obtained for the fundamental mode as well as for higher modes of vibration.

5. Concluding remarks. The value of the frequency of the fundamental mode of free vibration of a square plate clamped at its four edges has been calculated by using five coordinate-functions. The result is

$$(2\rho h a^4/D)p^2 = 81.54,$$

which should be compared with the exact value 80.9396 given several years ago by Tomotika [1]. It is to be noted that the error of our approximate value is less than 0.8 per cent. Also, it will be of worth mentioning that the calculation by use of three terms only still gives a fairly good approximate value of 81.63.

Further, a short account of the results obtained for the other two cases may serve to illustrate the accuracy of the present method.

The buckling load P for a square plate subject to uniform compressive forces at its four edges has been found to be given by

$$(a^2/D)P = 13.19.$$

This is very close to the value 13.086 which was obtained by Weinstein [3] and is known to be not larger than the true value.

A related problem of bending of a square plate under the action of uniform surface pressure, which has been discussed recently by the present writer [8] using a different method, can also be dealt with by the present method of analysis. It has been found that the maximum deflection $w_{\max}$ is given by

$$w_{\max}/(Qa^4/8D) = 0.161,$$

where Q is the pressure per unit area on one face of the plate. It will be seen that this is also in good agreement with the exact value 0.162, as given in Timoshenko's book [4].

Lastly, it will not be superfluous to remark that the last two numerical values have been obtained by making use of the same five coordinate-functions as in the case of vibration problem and that determinantal equations which yield these values have many terms in common with all the preceding three cases. These circumstances may illustrate the suitability of the present method for treating more complicated problems such as the vibration under the action of edge forces.

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THE STRESS DISTRIBUTION IN AN AEOLOTROPIC CIRCULAR DISK COMPRESSED DIAMETRICALLY

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The problem has been treated for a disk of an aeolotropic material which has two directions of symmetry at right angles in the plane of the plate.

In a previous paper, the writer had obtained an approximate solution of the problem [1]. The problem was, however, restricted to the cases where the direction of the compression coincides with one of the axes of symmetry of the material. Moreover, the solution is only valid for the particular case where the degree of the aeolotropy of the material is comparatively low.

Some years ago, the solution for an elliptic plate was given as a series in Faber's polynomials by Lechnitzky [2], but it seems that the boundary value problems are not treated in his paper, though the original description is inaccessible to the present writer, who owes his knowledge to the Zentralblatt für Mechanik, 6 (1938).

This is the re-analysis of the problem to avoid the deficiency of the previous solution, and an exact solution has been found out for a more general case, where the direction of the compression does not coincide with one of the axes of symmetry.

To solve the problem, a similar transformation of coordinates has been employed as was used formerly in the problem of an infinite aeolotropic plate with a circular hole [3].

When the material of the plate has the axes of symmetry at right angles in the (x, y) plane and these are taken to be parallel to the directions of x and y, then the differential equation for the stress function becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{k_1^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{k_2^2}\frac{\partial^2}{\partial y^2}\right)\chi = 0, \tag{1}$$

where $k_1^2 k_2^2 = S_{22}/S_{11}$, $k_1^2 + k_2^2 = (S_{66} + 2S_{12})/S_{11}$, and S_{11}, S_{22}, S_{12}, S_{66} are elastic constants characterizing the material.

The stresses for the case can generally be expressed in the forms [4]

$$\begin{aligned} X_x &= -R[k_1^2\phi_1(x + ik_1y) + k_2^2\phi_2(x + ik_2y)], \\ Y_y &= R[\phi_1(x + ik_1y) + \phi_2(x + ik_2y)], \\ X_y &= I[k_1\phi_1(x + ik_1y) + k_2\phi_2(x + ik_2y)], \end{aligned} \tag{2}$$

where ϕ_1 and ϕ_2 are two arbitrary functions, and R and I denote the real and imaginary parts of the complex function respectively.

We shall transform the Cartesian coordinates into the following systems of curvilinear coordinates

$$\begin{aligned} x + iy &= e^{\alpha+i\beta}, \\ x + ik_1y &= c' \cosh(\alpha' + i\beta'), \\ x + ik_2y &= c'' \cosh(\alpha'' + i\beta''), \end{aligned} \tag{3}$$

provided with

$$c' \cosh a' = 1, \quad c' \sinh a' = k_1, \quad c'' \cosh a'' = 1, \quad c'' \sinh a'' = k_2.$$

Then $\alpha = 0$, $\alpha' = a'$ and $\alpha'' = a''$ represent the circumference of a unit circle respectively, and moreover, at the boundary

$$\beta = \beta' = \beta''. \tag{4}$$

Now, we shall consider a disk compressed diametrically by a pair of opposite forces P acting on its periphery. When the direction of forces makes an angle θ with the direction of x, then the boundary condition becomes

$$\widehat{\alpha\alpha} = -(P/\pi)(1 + 2\sum_{n=1}^{\infty} \cos 2n(\beta - \theta)), \quad \widehat{\alpha\beta} = 0, \quad \text{at } \alpha = 0, \tag{5}$$

We consider two stress systems, each of which satisfies the boundary condition (6) or (7) respectively.

$$[\widehat{\alpha\alpha}]_1 = -(P/\pi)(1 + 2\sum_{n=1}^{\infty} \cos 2n\theta \cos 2n\beta), \quad [\widehat{\alpha\beta}]_1 = 0, \text{ at } \alpha = 0, \tag{6}$$

$$[\widehat{\alpha\alpha}]_2 = -(2P/\pi)\sum_{n=1}^{\infty} \sin 2n\theta \sin 2n\beta, \quad [\widehat{\alpha\beta}]_2 = 0, \text{ at } \alpha = 0. \tag{7}$$

Then adding these stress systems together, we have a stress system which satisfies the boundary condition (5).

First, we shall find a stress system which satisfies the boundary condition (6). From equation (2), we have

$$\begin{aligned}\widehat{\alpha\alpha} - \widehat{\beta\beta} - 2i\widehat{\alpha\beta} &= -e^{2i\beta}R[(1 + k_1^2)\phi_1(x + ik_1y) + (1 + k_2^2)\phi_2(x + ik_2y)] \\ &\quad -2ie^{2i\beta}I[k_1\phi_1(x + ik_1y) + k_2\phi_2(x + ik_2y)], \\ \widehat{\alpha\alpha} + \widehat{\beta\beta} &= R[(1 - k_1^2)\phi_1(x + ik_1y) + (1 - k_2^2)\phi_2(x + ik_2y)].\end{aligned} \tag{8}$$

Let us express the functions ϕ_1 and ϕ_2 in the forms

$$\begin{aligned}\phi_1(x + ik_1y) &= \sum_{n=1}^{\infty} A_n \cosh 2n(\alpha' + i\beta'), \\ \phi_2(x + ik_2y) &= \sum_{n=0}^{\infty} B_n \cosh 2n(\alpha'' + i\beta''),\end{aligned} \tag{9}$$

where A_n and B_n are real constants. Then the stresses can be expressed in the forms

$$\begin{aligned}[\widehat{\alpha\alpha}]_1 &= \tfrac{1}{2}\sum_{n=0}^{\infty} A_n\{(1 - k_1^2)\cosh 2n\alpha' \cos 2n\beta' \\ &\quad + 2k_1 \sinh 2n\alpha' \sin 2\beta \sin 2n\beta' - (1 + k_1^2)\cosh 2n\alpha' \cos 2\beta \cos 2n\beta'\} \\ &\quad + \tfrac{1}{2}\sum_{n=0}^{\infty} B_n\{(1 - k_2^2)\cosh 2n\alpha'' \cos 2n\beta'' \\ &\quad + 2k_2 \sinh 2n\alpha'' \sin 2\beta \sin 2n\beta'' - (1 + k_2^2)\cosh 2n\alpha'' \cos 2\beta \cos 2n\beta''\}, \\ [\widehat{\beta\beta}]_1 &= \tfrac{1}{2}\sum_{n=0}^{\infty} A_n\{(1 - k_1^2)\cosh 2n\alpha' \cos 2n\beta' \\ &\quad - 2k_1 \sinh 2n\alpha' \sin 2\beta \sin 2n\beta' + (1 + k_1^2)\cosh 2n\alpha' \cos 2\beta \cos 2n\beta'\} \\ &\quad + \tfrac{1}{2}\sum_{n=0}^{\infty} B_n\{(1 - k_2^2)\cosh 2n\alpha'' \cos 2n\beta'' \\ &\quad - 2k_2 \sinh 2n\alpha'' \sin 2\beta \sin 2n\beta'' + (1 + k_2^2)\cosh 2n\alpha'' \cos 2\beta \cos 2n\beta''\},\end{aligned} \tag{10}$$

$$[\widehat{\alpha\beta}]_1 = \tfrac{1}{2}\sum_{n=0}^{\infty} A_n\{(1+k_1^2)\cosh 2n\alpha' \sin 2\beta \cos 2n\beta' + 2k_1 \sinh 2n\alpha' \cos 2\beta \sin 2n\beta'\} + \tfrac{1}{2}\sum_{n=0}^{\infty} B_n\{(1+k_2^2)\cosh 2n\alpha'' \sin 2\beta \cos 2n\beta'' + 2k_2 \sinh 2n\alpha'' \cos 2\beta \sin 2n\beta''\}.$$

To satisfy the condition (6), the constants A_n, B_n have the following relations:

$$\begin{aligned}
&(1-k_1^2)(2A_0 - A_1) + (1-k_2^2)(2B_0 - B_1) = -4P/\pi,\\
&(1-k_1^2)\cosh 2a'(2A_1 - A_2 - 2A_0) + (1-k_2^2)\cosh 2a''(2B_1 - B_2 - 2B_0) = -(8P/\pi)\cos 2\theta,\\
&(1-k_1^2)\cosh 2n\alpha'(2A_n - A_{n+1} - A_{n-1}) + (1-k_2^2)\cosh 2na''(2B_n - B_{n+1} - B_{n-1}) = -(8P/\pi)\cos 2n\theta, \qquad n \geqq 2,\\
&(1-k_1^2)\cosh 2a'(2A_0 - A_2) + (1-k_2^2)\cosh 2a''(2B_0 - B_2) = 0,\\
&\qquad\qquad \cdots\cdots\cdots\\
&(1-k_1^2)\cosh 2na'(A_{n-1} - A_{n+1}) + (1-k_2^2)\cosh 2na''(B_{n-1} - B_{n+1}) = 0, \qquad n \geqq 2,
\end{aligned} \tag{11}$$

from which, we have

$$\begin{aligned}
&(1-k_1^2)\cosh 2na'A_n' + (1-k_2^2)\cosh 2na''B_n' = -(4P/\pi)\cos 2n\theta,\\
&(1-k_1^2)\cosh 2(n+1)a'A_n' + (1-k_2^2)\cosh 2(n+1)a''B_n' = (4P/\pi)\cos 2(n+1)\theta, \qquad n = 0, 1, 2, 3, \cdots
\end{aligned} \tag{12}$$

where

$$\left.\begin{aligned} A_0' &= 2A_0 - A_1,\\ B_0' &= 2B_0 - B_1, \end{aligned}\right| \quad \left.\begin{aligned} A_n' &= A_n - A_{n+1},\\ B_n' &= B_n - B_{n+1}, \end{aligned}\right| \qquad n \geqq 1.$$

Solving the simultaneous equations, we have

$$\begin{aligned}
A_n' &= -\frac{4P}{\pi(1-k_1^2)S_n}\{\cosh 2(n+1)a'' \cos 2n\theta + \cosh 2na'' \cos 2(n+1)\theta\},\\
B_n' &= \frac{4P}{\pi(1-k_2^2)S_n}\{\cosh 2(n+1)a' \cos 2n\theta + \cosh 2na' \cos 2(n+1)\theta\}, \qquad n = 0, 1, 2, 3, \cdots
\end{aligned} \tag{13}$$

where $S_n = \cosh 2na' \cosh 2(n+1)a'' - \cosh 2(n+1)a' \cosh 2na''$.
Hence the unknown constants A_n, B_n are found from the relations

$$\left.\begin{aligned} A_0 &= \tfrac{1}{2}\sum_{n=0}^{\infty} A_n',\\ B_0 &= \tfrac{1}{2}\sum_{n=0}^{\infty} B_n', \end{aligned}\right| \quad \left.\begin{aligned} A_m &= \sum_{n=m}^{\infty} A_n',\\ B_m &= \sum_{n=m}^{\infty} B_n', \end{aligned}\right| \qquad m \geqq 1. \tag{14}$$

Next, we shall find a stress system which satisfies the condition (7). Putting the arbitrary functions in the forms

$$\begin{aligned}\phi_1(x+ik_1y) &= -i\textstyle\sum_{n=0}^{\infty} C_n \cosh 2n(\alpha'+i\beta'),\\ \phi_2(x+ik_2y) &= -i\textstyle\sum_{n=1}^{\infty} D_n \cosh 2n(\alpha''+i\beta''),\end{aligned} \tag{15}$$

we have

$$\begin{aligned}[\widehat{\alpha\alpha}]_2 &= \tfrac{1}{2}\textstyle\sum_{n=0}^{\infty} C_n\{(1-k_1^2)\sinh 2n\alpha' \sin 2n\beta'\\ &\quad - 2k_1\cosh 2n\alpha' \sin 2\beta\cos 2n\beta' - (1+k_1^2)\sinh 2n\alpha'\cos 2\beta\sin 2n\beta'\}\\ &\quad + \tfrac{1}{2}\textstyle\sum_{n=1}^{\infty} D_n\{(1-k_2^2)\sinh 2n\alpha'' \sin 2n\beta''\\ &\quad - 2k_2\cosh 2n\alpha'' \sin 2\beta\cos 2n\beta'' - (1+k_2^2)\sinh 2n\alpha''\cos 2\beta\sin 2n\beta''\},\\ [\widehat{\beta\beta}]_2 &= \tfrac{1}{2}\textstyle\sum_{n=0}^{\infty} C_n\{(1-k_1^2)\sinh 2n\alpha' \sin 2n\beta'\\ &\quad + 2k_1\cosh 2n\alpha' \sin 2\beta\cos 2n\beta' + (1+k_1^2)\sinh 2n\alpha'\cos 2\beta\sin 2n\beta'\}\\ &\quad + \tfrac{1}{2}\textstyle\sum_{n=1}^{\infty} D_n\{(1-k_2^2)\sinh 2n\alpha'' \sin 2n\beta''\\ &\quad + 2k_2\cosh 2n\alpha'' \sin 2\beta\cos 2n\beta'' + (1+k_2^2)\sinh 2n\alpha''\cos 2\beta\sin 2n\beta''\},\\ [\widehat{\alpha\beta}]_2 &= \tfrac{1}{2}\textstyle\sum_{n=0}^{\infty} C_n\{(1+k_1^2)\sinh 2n\alpha' \sin 2\beta\sin 2n\beta'\\ &\quad - 2k_1\cosh 2n\alpha' \cos 2\beta\cos 2n\beta'\}\\ &\quad + \tfrac{1}{2}\textstyle\sum_{n=1}^{\infty} D_n\{(1+k_2^2)\sinh 2n\alpha'' \sin 2\beta\sin 2n\beta''\\ &\quad - 2k_2\cosh 2n\alpha'' \cos 2\beta\cos 2n\beta''\}.\end{aligned} \tag{16}$$

To satisfy the condition (7), we have similar relations between the unknown constants C_n and D_n as in the former case, and from which we can find

$$\begin{aligned}&2(1-k_1^2)\sinh 2a'.\ C_0 = (1-k_1^2)\sinh 2a'.\ C_1 + (1-k_2^2)\sinh 2a''.\\ &\qquad D_1 + (4P/\pi)\sin 2\theta,\\ &C_m = \textstyle\sum_{n=m}^{\infty} C'_n, \qquad D_m = \sum_{n=m}^{\infty} D'_n, \qquad m \geqq 1,\end{aligned} \tag{17}$$

where

$$\left.\begin{aligned}C'_n &= -\frac{4P}{\pi(1-k_1^2)S'_n}\\ &\quad\cdot\{\sinh 2(n+1)a''\sin 2n\theta + \sinh 2na''\sin 2(n+1)\theta\},\\ D'_n &= \frac{4P}{\pi(1-k_2^2)S'_n}\\ &\quad\cdot\{\sinh 2na'\sin 2(n+1)\theta + \sinh 2(n+1)a'\sin 2n\theta\},\end{aligned}\right\} \quad n \geqq 1,$$

and

$$S'_n = \sinh 2na'\sinh 2(n+1)a'' - \sinh 2(n+1)a'\sinh 2na''.$$

Now we have the stress system which satisfies the boundary condition (5) in the forms

$$\widehat{\alpha\alpha} = [\widehat{\alpha\alpha}]_1 + [\widehat{\alpha\alpha}]_2, \qquad \widehat{\beta\beta} = [\widehat{\beta\beta}]_1 + [\widehat{\beta\beta}]_2, \qquad \widehat{\alpha\beta} = [\widehat{\alpha\beta}]_1 + [\widehat{\alpha\beta}]_2. \tag{18}$$

As an example, the stress distribution in a disk of oak will be evaluated numerically, because wooden plates possess a very high degree of aeolotropy. The plate is assumed to be cut parallel to the grain of the wood. The axis of x is taken parallel to the direction of the fibres, and the axis of y is taken in the direction perpendicular to it and also to all lines which passes through the central core of the tree. Then the values of k_1 and k_2 for the case become [5]

$$k_1 = 2{,}307, \qquad k_2 = 1{,}053.$$

The corresponding values of constants, calculated from equations (14) and (17), for three different cases of the direction of compression ($\theta = 0, \pi/2, \pi/4$) are tabulated in Tables 1 and 2.

TABLE 1
(*Unit* P/π)

n	$\theta = 0$		$\theta = \pi/2$	
	A_n	B_n	A_n	B_n
0	0,2520	−0,4433	1,129	−2,613
1	−0,4454	0,6264 × 10^{-1}	1,205	−0,1734
2	0,2088	−0,1947 × 10^{-2}	0,5161	−0,4828 × 10^{-2}
3	−0,8492 × 10^{-1}	0,5182 × 10^{-4}	0,2068	−0,1263 × 10^{-3}
4	0,3370 × 10^{-1}	−0,1344 × 10^{-5}	0,8191 × 10^{-1}	−0,3267 × 10^{-5}
5	−0,1334 × 10^{-1}	0,3473 × 10^{-7}	0,3238 × 10^{-1}	−0,8438 × 10^{-7}
6	0,5269 × 10^{-2}	−0,8966 × 10^{-9}	0,1280 × 10^{-1}	−0,2178 × 10^{-8}
7	−0,2083 × 10^{-2}	0,2315 × 10^{-10}	0,5060 × 10^{-2}	−0,5622 × 10^{-10}
8	0,8233 × 10^{-3}	−0,5978 × 10^{-12}	0,2000 × 10^{-2}	−0,1451 × 10^{-11}
9	−0,3252 × 10^{-3}	0,1543 × 10^{-13}	0,7902 × 10^{-3}	−0,3748 × 10^{-13}
10	0,1286 × 10^{-3}	−0,3981 × 10^{-15}	0,3123 × 10^{-3}	−0,9676 × 10^{-15}
11	−0,5082 × 10^{-4}	0,1028 × 10^{-16}	0,1234 × 10^{-3}	−0,2497 × 10^{-16}
12	0,2008 × 10^{-4}	−0,2654 × 10^{-18}	0,4878 × 10^{-4}	−0,6448 × 10^{-18}

TABLE 2
(*Unit* P/π)

n	$\theta = \pi/4$			
	A_n	B_n	C_n	D_n
0	0,3633	−1,473	0,0563	
1	−0,2744	0,5533 × 10^{-1}	−0,8244	0,1536
2	−0,2571	0,3314 × 10^{-2}	0,1133	−0,1443 × 10^{-2}
3	0,4447 × 10^{-1}	−0,3719 × 10^{-4}	0,1051	−0,8775 × 10^{-4}
4	0,4133 × 10^{-1}	−0,2256 × 10^{-5}	−0,1760 × 10^{-1}	0,9604 × 10^{-6}
5	−0,6954 × 10^{-2}	0,2478 × 10^{-7}	−0,1635 × 10^{-1}	0,5829 × 10^{-7}
6	−0,6461 × 10^{-2}	0,1504 × 10^{-8}	0,2749 × 10^{-2}	−0,6398 × 10^{-9}
7	0,1086 × 10^{-2}	−0,1652 × 10^{-10}	0,2554 × 10^{-2}	−0,3883 × 10^{-10}
8	0,1009 × 10^{-3}	−0,1003 × 10^{-11}	−0,4202 × 10^{-3}	0,4266 × 10^{-12}
9	−0,1697 × 10^{-3}	0,1101 × 10^{-13}	−0,3988 × 10^{-3}	0,2589 × 10^{-13}
10	−0,1577 × 10^{-3}	0,6684 × 10^{-15}	0,6706 × 10^{-4}	−0,2843 × 10^{-15}
11	0,2650 × 10^{-4}	−0,7338 × 10^{-17}	0,6231 × 10^{-4}	−0,1725 × 10^{-16}
12	0,2462 × 10^{-4}	−0,4454 × 10^{-18}	−0,1047 × 10^{-4}	0,1894 × 10^{-18}

From equations (10) and (16), the stress distributions on the radii, parallel and perpendicular to the direction of compression are obtained for these cases and are shown in Figs. 1, 2 and 3 respectively. For the convenience of comparison,

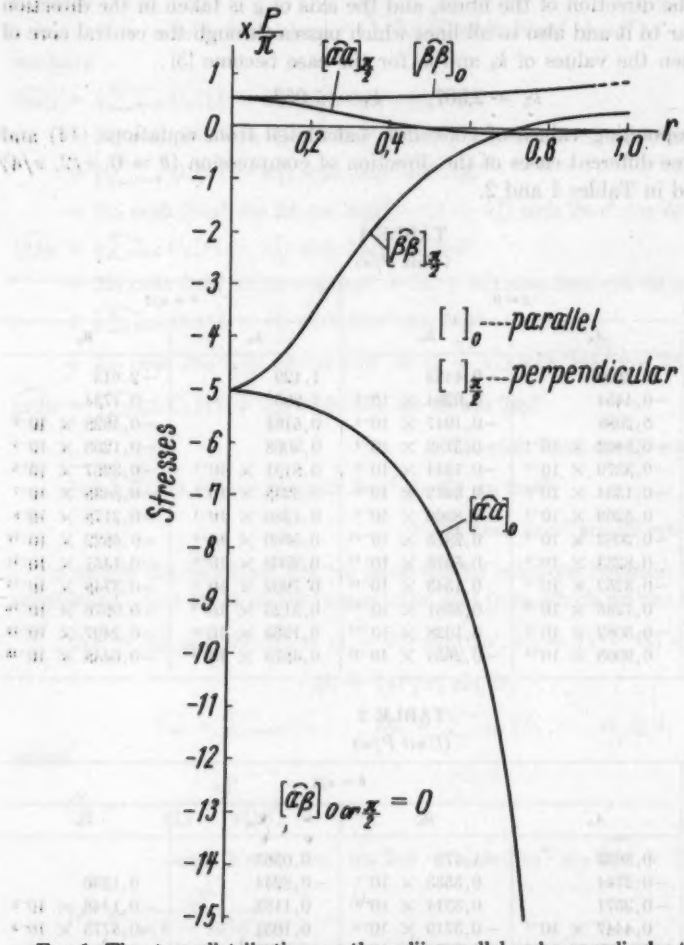

FIG. 1. The stress distributions on the radii, parallel and perpendicular to the direction of compression, when $\theta = 0$.

the stress distributions for the isotropy are also shown by curves in Fig. 5. From Figs. 3 and 4, we see that the stresses for $\theta = \pi/4$ distribute almost similarly on the radii, compared as those of the isotropy except the shearing stress.

Formerly the writer had obtained the stress distribution in an aeolotropic

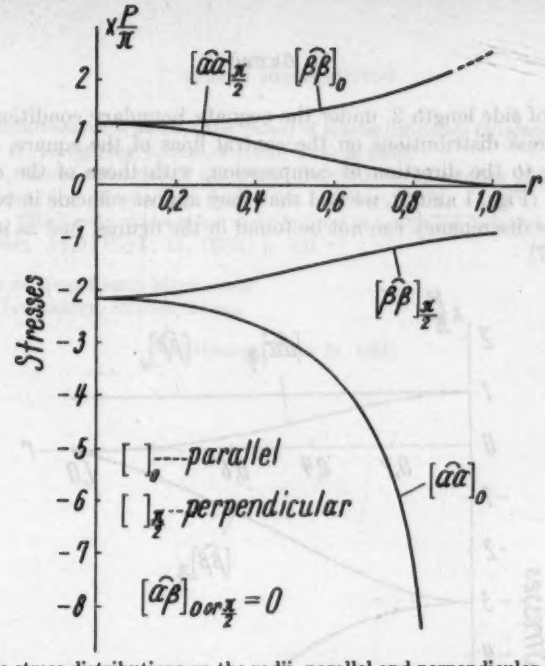

Fig. 2. The stress distributions on the radii, parallel and perpendicular to the direction of compression, when $\theta = \pi/2$.

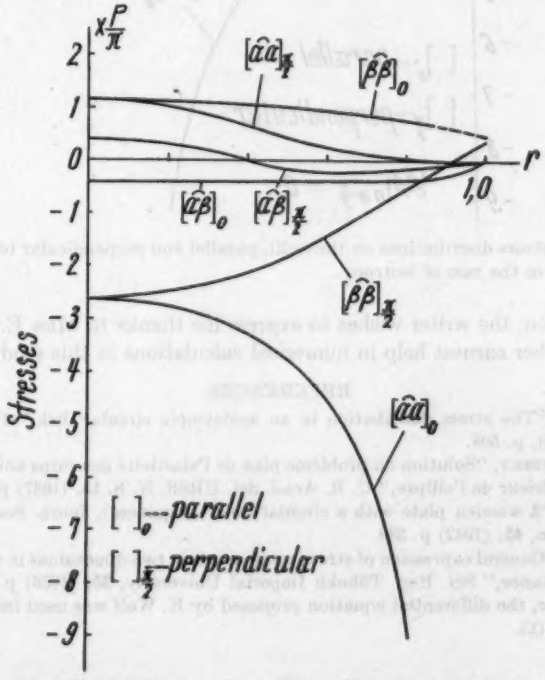

Fig. 3. The stress distributions on the radii, parallel and perpendicular to the direction of compression, when $\theta = \pi/4$.

square plate of side length 2, under the cognate boundary conditions [6]. Comparing the stress distributions on the central lines of the square, parallel and perpendicular to the direction of compression, with those of the circular disk obtained here (Figs. 1 and 2), we find that they almost coincide in two cases and an appreciable discrepancy can not be found in the figures, just as in the case of the isotropy [7].

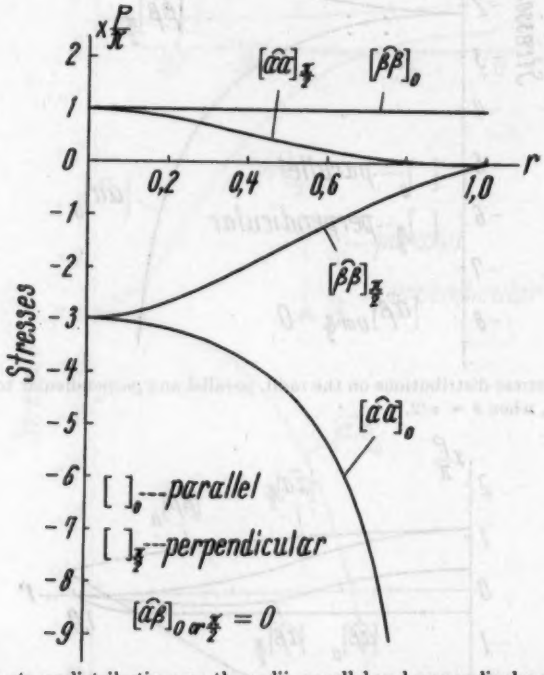

FIG. 4. The stress distributions on the radii, parallel and perpendicular to the direction of compression in the case of isotropy.

In conclusion, the writer wishes to express his thanks to Miss E. Itagaki, his assistant, for her earnest help in numerical calculations in this study.

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FINITE ELASTIC-PLASTIC TORSION

By B. R. Seth

1. Introduction. In the customary treatment of elastic-plastic torsion the displacements and the strains are both assumed to be small. In a number of plastic problems both these assumptions need not hold good. We propose to discuss the torsion of a circular cylinder by taking finite components of displacements and strain. The problem is examined from the point of view of both the rival theories of plasticity—the theory of plastic flow and the theory of plastic deformation. Recently P. G. Hodge [1] has discussed the torsion problem for large values of α, the angle of twist. He uses small strains and the deformation theory and compares his approximate results with those of V. V. Sokolowsky [2].

We propose to show that, for a first approximation, in which axial stresses are neglected, both these theories give similar results for the torsion of a circular cylinder. This verifies the general result obtained by H. Geringer and W. Prager, [3] for any section. For a second approximation finite components of displacements and strains are taken. The corresponding results give a radial displacement in the plastic region, not given by the ordinary theory. If the deformation theory is used, the axial stresses in the elastic and plastic domains are found to be of order α^2 and α respectively, and hence the radial stress cannot satisfy the continuity condition on the elastic-plastic boundary. This indicates that the discontinuities in stresses encountered in the discussion of some plastic problems (not torsion problem) may have their origin in the use of small deformations and the deformation theory in which the co-ordinates used refer to a point in the unstrained state and not to a point in the strained state at any instant. The importance of using the strained framework, otherwise called Eulerian co-ordinates, is now being increasingly appreciated by workers on plastic problems [4].

2. First approximation-deformation theory. In this case small displacements and strains are used. We know that the displacements are given by

$$u_r = 0, \qquad u_\theta = \alpha r z, \qquad w = 0, \tag{2.1}$$

and the strains by

$$e_{rr}, e_{\theta\theta}, e_{zz}, e_{rz}, e_{r\theta} = 0,$$
$$e_{\theta z} = \alpha r. \tag{2.2}$$

The only non-vanishing stress is $\tau_{\theta z}$ which is given by

$$e_{\theta z} = 2\lambda_1 \tau_{\theta z}, \tag{2.3}$$

λ_1 being a function of r and α, still to be determined. The yield condition gives

$$\tau_{\theta z} = Y/\sqrt{3} = \alpha r/2\lambda_1 \tag{2.4}$$

so that

$$\lambda_1 = \alpha r\sqrt{3}/2Y. \tag{2.5}$$

Over the elastic-plastic boundary $r = a$ (See Fig. 1)

$$\tau_{\theta z} = \mu\alpha r = \mu\alpha a = Y/\sqrt{3},$$

and hence a is given by

$$a = Y/(\mu\alpha\sqrt{3}). \tag{2.6}$$

The torsional couple is seen to be

$$M = (\pi/6\sqrt{3})\,[3\sqrt{3}\mu\alpha a^4 + 4Y(b^3 - a^3)]. \tag{2.7}$$

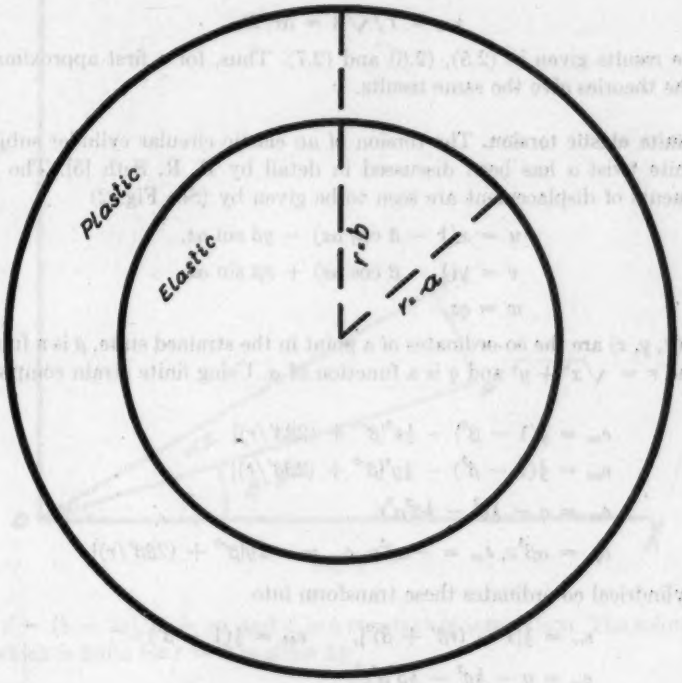

FIG. 1

3. First approximation—flow theory. In this case displacements and strains remain the same as given in (2.1) and (2.2). The stress-strain relations of the deformation theory given by equations of the type

$$e_{rr} - e = \lambda_1(\tau_{rr} - \tau), \qquad e_{\theta z} = 2\lambda_1\tau_{\theta z} \tag{3.1}$$

where

$$e = \tfrac{1}{3}(e_{rr} + e_{\theta\theta} + e_{zz}), \qquad \tau = \tfrac{1}{3}(\tau_{rr} + \tau_{\theta\theta} + \tau_{zz}),$$

now take the form

$$\dot{e}_{rr} - \dot{e} = (1/\mu)(\dot{\tau}_{rr} - \tau) + \lambda_1(\tau_{rr} - \tau),$$
$$\dot{e}_{\theta z} = (1/\mu)\dot{\tau}_{\theta z} + 2\lambda_1\tau_{\theta z}. \tag{3.2}$$

Here the dot may be taken to indicate differentiation with respect to any monotonically increasing parameter of the plastic flow. In our case we shall find it convenient to take this parameter as the logarithm of the angle of twist per unit length.

Thus we get

$$\dot{e}_{\theta z} = \partial\alpha r/\partial \log \alpha = \alpha r, \tag{3.3}$$

$$\tau_{\theta z} = Y/\sqrt{3} = \alpha r/2\lambda_1 \tag{3.4}$$

and the results given in (2.5), (2.6) and (2.7). Thus, for a first approximation, both the theories give the same results.

4. Finite elastic torsion. The torsion of an elastic circular cylinder subjected to a finite twist α has been discussed in detail by B. R. Seth [5]. The finite components of displacement are seen to be given by (See Fig. 2)

$$u = x(1 - \beta\cos\alpha z) - y\beta\sin\alpha z,$$
$$v = y(1 - \beta\cos\alpha z) + x\beta\sin\alpha z, \tag{4.1}$$
$$w = qz.$$

where (x, y, z) are the co-ordinates of a point in the strained state, β is a function of α and $r = \sqrt{x^2 + y^2}$ and q is a function of α. Using finite strain components we get

$$e_{xx} = \tfrac{1}{2}(1 - \beta^2) - \tfrac{1}{2}x^2[\beta'^2 + (2\beta\beta'/r)]$$
$$e_{yy} = \tfrac{1}{2}(1 - \beta^2) - \tfrac{1}{2}y^2[\beta'^2 + (2\beta\beta'/r)]$$
$$e_{zz} = q - \tfrac{1}{2}q^2 - \tfrac{1}{2}\beta^2\alpha^2r$$
$$e_{yz} = \alpha\beta^2x,\ e_{zx} = -\alpha\beta^2y,\ e_{xy} = -xy[\beta'^2 + (2\beta\beta'/r)].$$

In cylindrical co-ordinates these transform into

$$e_{rr} = \tfrac{1}{2}[1 - (r\beta' + \beta)^2], \qquad e_{\theta\theta} = \tfrac{1}{2}(1 - \beta^2)$$
$$e_{zz} = q - \tfrac{1}{2}q'^2 - \tfrac{1}{2}\beta^2\alpha^2r^2, \tag{4.2}$$
$$e_{r\theta} = 0, \qquad e_{rz} = 0, \qquad e_{\theta z} = \alpha\beta^2r,$$

dashes denoting differentiation with respect to r.*

The corresponding stresses in cylindrical co-ordinates are

$$\tau_{rr} = 3\lambda e + \mu - \mu(r\beta' + \beta)^2, \qquad \tau_{\theta\theta} = 3\lambda e + \mu(1 - \beta^2),$$
$$\tau_{zz} = 3\lambda e + \mu(2q - q^2 - \alpha^2\beta^2r^2), \tag{4.31}$$
$$\tau_{rz} = 0, \qquad \tau_{r\theta} = 0, \qquad \tau_{\theta z} = \mu\alpha\beta^2r,$$

* These may be directly obtained from the mixed tensor forms of the strain components [6].

where

$$3e = 1 - \tfrac{1}{2}\beta^2 - \tfrac{1}{2}(r\beta' + \beta)^2 + q - \tfrac{1}{2}q^2 - \tfrac{1}{2}\beta^2\alpha^2r^2. \tag{4.32}$$

These satisfy the equations of equilibrium if β satisfies the differential equation

$$[1 + (1 - d)\alpha^2r^2]\beta^2 + (r\beta' + \beta)^2 + d\int r\beta'^2\,dr = L, \tag{4.4}$$

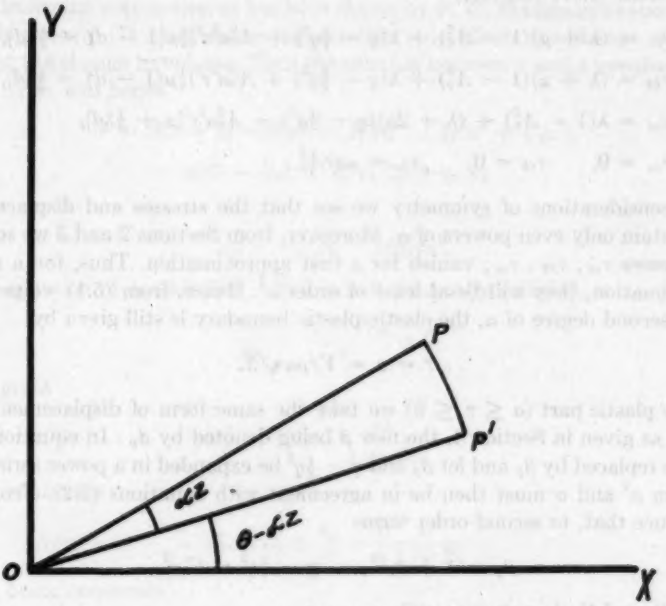

FIG. 2

where $d = (1 - 2\sigma)/(1 - \sigma)$, and L is a constant of integration. The solution of (4.4) which is finite for $r = 0$ is given by

$$\beta = A_0\left[1 - \tfrac{1}{4}(1 - d)\alpha^2r^2 + \frac{6 - d}{12.8^2}(1 - d)^2\alpha^4r^4 - \frac{d^2}{9.8^4}(1 - d)^4\alpha^6r^6 + \cdots\right]. \tag{4.5}$$

It is shown that this solution is convergent at least for $\alpha r \leq 1$. In the elastic solution we find

$$A_0 = 1 + \tfrac{1}{4}(1 - d)\alpha^2a^2 + \tfrac{1}{8.12}(1 - d)^2\alpha^4a^4 + \cdots. \tag{4.51}$$

5. Finite elastic-plastic torsion. For the elastic part ($0 \leq r \leq a$) shown in Fig. 1 the displacements, strains and stresses are those given in Section 4. The con-

stants A_0 and q are to be determined from the boundary conditions over $r = a$ and the plane ends.

The elastic-plastic boundary $r = a$ is to be determined from the yield condition

$$(\tau_{rr} - \tau_{\theta\theta})^2 + (\tau_{\theta\theta} - \tau_{zz})^2 + (\tau_{zz} - \tau_{rr})^2 + 6\tau_{\theta z}^2 = 2Y^2, \tag{5.1}$$

Y being the yield stress in tension. If we neglect powers of α higher than the second we get from (4.31) and (4.5)

$$\begin{aligned}
\tau_{rr} &= (\lambda + \mu)(1 - A_0^2) + \lambda(q - \tfrac{1}{2}q^2) + A_0^2\alpha^2r^2[\tfrac{3}{4}\mu(1 - d) - \tfrac{1}{2}\lambda d], \\
\tau_{\theta\theta} &= (\lambda + \mu)(1 - A_0^2) + \lambda(q - \tfrac{1}{2}q^2) + A_0^2\alpha^2r^2[\tfrac{1}{4}\mu(1 - d) - \tfrac{1}{2}\lambda d], \\
\tau_{zz} &= \lambda(1 - A_0^2) + (\lambda + 2\mu)(q - \tfrac{1}{2}q^2) - A_0^2\alpha^2r^2[\mu + \tfrac{1}{2}\lambda d], \\
\tau_{rz} &= 0, \qquad \tau_{r\theta} = 0, \qquad \tau_{\theta z} = \mu\alpha r A_0^2 .
\end{aligned} \tag{5.2}$$

From considerations of symmetry we see that the stresses and displacements can contain only even powers of α. Moreover, from Sections 2 and 3 we see that the stresses τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, vanish for a first approximation. Thus, for a second approximation, they will be at least of order α^2. Hence, from (5.1) we get that, to the second degree of α, the elastic-plastic boundary is still given by

$$r = a = Y/\mu\alpha\sqrt{3}. \tag{5.3}$$

For the plastic part ($a \leq r \leq b$) we take the same form of displacements and strains as given in Section 4, the new β being denoted by β_p. In equation (4.2) let β be replaced by β_p and let β_p and $q - \frac{1}{2}q^2$ be expanded in a power series. The terms in α^0 and α must then be in agreement with equations (2.2). From this we deduce that, to second order terms

$$\beta_p = 1 + \beta_1\alpha^2, \qquad q - \tfrac{1}{2}q^2 = C_1\alpha^2, \tag{5.4}$$

where B_1 and C_1 do not vary with α.
Proceeding as in Section 3, we get

$$\begin{aligned}
\dot{e}_{rr} - \dot{e} &= -\tfrac{1}{3}(2B_1 + 4B_1'r - r^2 + 2C_1)\alpha^2, \\
\dot{e}_{\theta\theta} - \dot{e} &= -\tfrac{1}{3}(2B_1 - 2B_1'r - r^2 + 2C_1)\alpha^2, \\
\dot{e}_{zz} - \dot{e} &= \tfrac{1}{3}(4B_1 + 2B_1'r - 2r^2 + 4C_1)\alpha^2, \\
\dot{e}_{\theta z} &= (1 + 2B_1\alpha^2)r\alpha,
\end{aligned} \tag{5.5}$$

dashes denoting differentiation with respect to r, and dots with respect to $\log \alpha$.

The first order of value of λ_1 in (3.1) and (3.2) has been found to be $\frac{1}{2}\alpha r\sqrt{3}/Y$. If we use the deformation theory and substitute it in (3.1), we see that the radial stresses in the plastic domain must be of order α, while those in the elastic domain are known to be of order α^2. Thus we cannot satisfy the condition of the continuity of τ_{rr} over the elastic-plastic boundary.

If we use the flow theory we get from (3.2)

$$\tau_{rr} - \tau = A[1 - (\mu\alpha\sqrt{3}/Y)r] - \tfrac{1}{2}\mu(2B_1 + 4B_1'r - r^2 + 2C_1)\alpha^2 + \cdots$$

A being a constant of integration.

As e is of the type $D_0 + D_1\alpha^2$, D_0, D_1 being independent of α, we see that for satisfying the conditions of continuity of τ_{rr} over $r = a$ we must have $A = 0$, as we do in the first approximation. The values of τ_{rr} in the elastic and plastic domains are now both of order α^2 and can be equated.

To determine τ we notice, as has been shown by P. W. Bridgman's experiments at high pressure [7], that plastic deformation only effects the change of shape and not the change in volume. Thus the relation between τ and e remains of the elastic type, and hence

$$\tau = 3ke = k[-2B_1\alpha^2 - B_1'r\alpha^2 - \tfrac{1}{2}r^2\alpha^2 + C_1\alpha^2],$$

$$3k(1 - 2\sigma) = E = 2\mu(1 + \sigma). \tag{5.6}$$

Also

$$\tau_{rr} - \tau_{\theta\theta} = -2\mu B_1'r\alpha^2.$$

The equation of equilibrium, which is not identically satisfied, is

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0.$$

which gives

$$rB_1'' + 3B_1' + \frac{r\lambda}{\lambda + 2\mu} = 0,$$

whose solution is

$$B_1 = k_0 + \frac{k_1}{r^2} - \frac{1}{8}\frac{\lambda r^2}{\lambda + 2\mu} \tag{5.7}$$

K_0, K_1 being constants.

This shows that in a fully plastic state the axis of the cylinder may be a line of singularity. Since $\beta = \beta_p$ over $r = a$ we see from (4.5) that we can write

$$\begin{aligned} \beta &= 1 + \left(K_0 + \frac{K_1}{a^2} - \frac{1}{8}\frac{\lambda r^2}{\lambda + 2\mu}\right)\alpha^2, \\ \beta_p &= 1 + \left(K_0 + \frac{K_1}{r^2} - \frac{1}{8}\frac{\lambda r^2}{\lambda + 2\mu}\right)\alpha^2. \end{aligned} \tag{5.8}$$

The axial stresses in the plastic domain are given by

$$\begin{aligned} \tau_{rr} &= \left[\lambda C_1 - 2K_0(\lambda + \mu) + \frac{2\mu K_1}{r^2} - \frac{\lambda\mu r^2}{4(\lambda + 2\mu)}\right]\alpha^2, \\ \tau_{\theta\theta} &= \left[\lambda C_1 - 2K_0(\lambda + \mu) - \frac{2\mu K_1}{r^2} - \frac{3\lambda\mu r^2}{4(\lambda + 2\mu)}\right]\alpha^2, \\ \tau_{zz} &= \left[(\lambda + 2\mu)C_1 - 2K_0\lambda - \frac{2\mu(\lambda + \mu)r^2}{\lambda + 2\mu}\right]\alpha^2. \end{aligned} \tag{5.81}$$

The axial stresses in the elastic domain are

$$\tau_{rr} = \left[\lambda C_1 - 2K_0(\lambda+\mu) - \frac{2(\lambda+\mu)K_1}{r^2} - \frac{\lambda\mu r^2}{4(\lambda+2\mu)}\right]\alpha^2,$$

$$\tau_{\theta\theta} = \left[\lambda C_1 - 2K_0(\lambda+\mu) - \frac{2(\lambda+\mu)K_1}{a^2} - \frac{3\lambda\mu r^2}{4(\lambda+2\mu)}\right]\alpha^2,$$

$$\tau_{zz} = \left[(\lambda+2\mu)C_1 - 2K_0\lambda - \frac{2\lambda K_1}{a^2} - \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}r^2\right]\alpha^2.$$

But these stresses are to be continuous over the elastic-plastic boundry $r = a$. This shows that $K_1 = 0$.

Again $\tau_{rr} = 0$ over $r = b$, and thus we get

$$\lambda C_1 - 2K_0(\lambda+\mu) - [\lambda\mu b^2/4(\lambda+2\mu)] = 0. \tag{5.82}$$

The condition to be satisfied over the plane ends is

$$\int_0^b 2\pi r.\tau_{zz}\,dr = 0,$$

which gives

$$(\lambda+2\mu)C_1 - 2\lambda K_0 - \mu[(\lambda+\mu)/(\lambda+2\mu)]b^2 = 0. \tag{5.83}$$

The two equations (5.82) and (5.83) give

$$C_1 = \tfrac{1}{4}b^2,\ K_0 = \tfrac{1}{8}b^2\lambda/(\lambda+2\mu).$$

With these values the stresses become

$$\tau_{rr} = \frac{1}{4}\frac{\lambda\mu}{\lambda+2\mu}(b^2-r^2)\alpha^2, \qquad \tau_{\theta\theta} = \frac{1}{4}\frac{\lambda\mu}{\lambda+2\mu}(b^2-3r^2)\alpha^2,$$

$$\tau_{zz} = \frac{\mu(\lambda+\mu)}{\lambda+2\mu}(b^2-2r^2)\alpha^2, \qquad \tau_{\theta z} = \mu r\alpha \text{ (elastic)}, \tag{5.9}$$

$$\tau_{\theta z} = Y/\sqrt{3} \text{ (plastic)}, \qquad \tau_{r\theta} = 0, \qquad \tau_{rz} = 0.$$

To the second order of α the torsional couple remains the same as given in (2.7).

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A SOLUTION OF THE TELEGRAPH EQUATION WITH APPLICATION TO TWO DIMENSIONAL SUPERSONIC SHEAR FLOW

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Introduction. The Riemann method of integration[1, 2] is a well-known method of arriving at an integral solution of the two dimensional linear hyperbolic partial differential equation when the Cauchy data along a non-characteristic curve is properly specified. Recently, in the treatment of supersonic shear flow over a two dimensional thin airfoil, the problem of obtaining a solution to the telegraph equation has arisen. However, it has been found that for this problem a portion of the Cauchy data becomes precisely the unknown with which we are concerned. As a result, when the Riemann formulation is adopted, the integral relation becomes Volterra's integral equation of the first kind. To solve this integral equation the technique of the Laplace transform[3] is applied. Unfortunately, branch points occurring in the inversion formula make the integral difficult to evaluate. Consequently, the general solution is too complicated to derive. However, for the particular case of supersonic shear flow under consideration,* a special expansion has been found by which the given normal derivative may be represented and which will cancel out the factors which introduce the branch points in the inversion formula. An explicit expression may then be obtained. Some examples have been given to illustrate the effects of shear flow over a wedge and a parabolic arc profile.

The telegraph equation and the corresponding initial and boundary conditions. The telegraph equation with which we are concerned may be written in the general form

$$a\partial^2 u/\partial t^2 + 2b\partial u/\partial t + cu = \partial^2 u/\partial x^2, \qquad a > 0 \qquad (1)$$

a, b, and c being constant coefficients.

A solution of Eq. (1) is sought which satisfies the boundary conditions

$$(u_t)_{t=0} = f(x) \qquad 0 \leqq x \leqq 1 \qquad (2a)$$

$$u = u_t = u_x = 0 \qquad x - (1/\sqrt{a})t < 0 \qquad (2b)$$

These boundary conditions determine a unique solution in the region $t \geqq 0$, $x - (1/\sqrt{a})t \leqq 1$ and are typical of the problem of supersonc flow over a two dimensional body whose profile alone is specified.

Since Eq. (2a) is concerned with the derivative of u, it is found convenient to differentiate Eq. (1) with respect to t and to solve directly for $u_t(x, t)$ instead of for u. Subsequent integration with respect to t will then give the desired solution $u(x, t)$. Differentiating and introducing the change of dependent variable

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$$u_t = e^{-(b/a)t}u'_t \tag{3}$$

an equation is obtained in which the first partial derivative of the dependent variable u'_t does not appear:

$$A^2\partial^2 u'_t/\partial x^2 - \partial^2 u'_t/\partial t^2 + B^2 u'_t = 0 \tag{4}$$

where

$$A^2 = 1/a, \qquad B^2 = (b^2 - ac)/a^2$$

The corresponding boundary conditions are

$$(u'_t)_{t=0} = f(x) \qquad 0 \leqq x \leqq 1 \tag{4a}$$

$$u' = u'_t = u'_x = 0 \qquad x - At < 0 \tag{4b}$$

Solution by the Riemann method. A solution of Eq. (4) may be obtained by the Riemann method in a manner similar to that devised by du Bois Reymond and also by Picard.[1] To apply this method to the problem at hand we must first note that instead of Cauchy data u, u_t along a single non-characteristic line segment, we are given u_t on the x-axis for $0 \leqq x \leqq 1$ and u, u_t along any line through the origin and ahead of the characteristic line $x - At = 0$. In other words, the data is given along two intersecting lines which are separated by a characteristic line. The value of $u(x, 0)$ for $0 \leqq x \leqq 1$ is an unknown for which we eventually wish to solve. Accordingly, the contour of integration, as shown in Figure (1), is made up of the two data lines and the two characteristic lines PA and PB which pass through the point of interest $P(x, t)$. The Riemann method[1] then gives the solution of Eq. (4) as

$$\begin{aligned} 2u'_t(x, t) &= u'_t(x - At, 0) \\ &+ (1/A)\int_{0_-}^{x-At} u'_{tt}(\xi, 0)J_0[(B/A)\sqrt{(x-\xi)^2 - A^2t^2}]\,d\xi \\ &+ Bt\int_{0_-}^{x-At} u'_t(\xi, 0)\frac{J'_0[(B/A)\sqrt{(x-\xi)^2 - A^2t^2}]}{\sqrt{(x-\xi)^2 - A^2t^2}}\,d\xi \end{aligned} \tag{5}$$

Since $u'_{tt}(\xi, 0)$ is not actually known explicitly, Eq. (5) does not yet represent the final solution. Instead it is an integral relationship between $u'_t(x, t)$ at any point $P(x, t)$ and $u'_{tt}(\xi, 0)$ on the boundary.

Determination of $u'_{tt}(\xi, 0)$. To determine $u'_{tt}(\xi, 0)$ let us put $t = 0$ in Eq. (5). Then, denoting $u'_{tt}(\xi, 0)$ by $g(\xi)$, we have

$$Au'_t(x, 0) = \int_{0_-}^{x} g(\xi)J_0[(B/A)(x - \xi)]\,d\xi \tag{6}$$

Eq. (6) is Volterra's integral equation of the first kind for the function $g(\xi)$. The solution of Eq. (6) may be obtained with the aid of the Laplace transform[2] after normalization as follows:

Let

$$\xi' = (B/A)\xi, \qquad x' = (B/A)x, \qquad g_1(\xi') = g(\xi)/(B/A)$$

after which we have

$$Au_i'(x', 0) = \int_{0-}^{x'} g_1(\xi')J_0(x' - \xi')\, d\xi' \tag{7}$$

The Laplace transform of $F(x)$ is defined in the usual manner as

$$L[F(x)] = s\int_0^\infty F(x)e^{-sx}\, dx$$

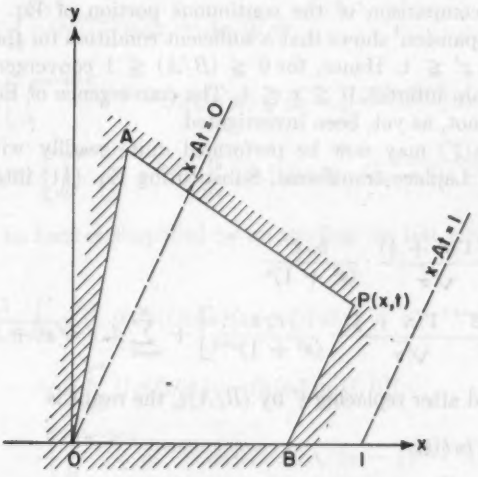

FIG. 1. Region of integration

Taking the transform of Eq. (7), we have by virtue of the convolution theorem and after transposition:

$$L[g_1(x')] = \sqrt{s^2 + 1}\cdot L[Au_i'(x', 0)] \tag{8}$$

The inversion of Eq. (8) is

$$g_1(\xi') = \int_{k-i\infty}^{k+i\infty} L[Au_i'(\xi', 0)]\sqrt{s^2 + 1}\, e^{st'}\, ds/s, \qquad k > 0 \tag{9}$$

The integral in Eq. (9) is difficult to evaluate because of the two branch points at $s = \pm i$. However, this integral may be evaluated if the given boundary condition, $Au_i'(x', 0)$, is expanded in such a way that its transform $L[Au_i'(x', 0)]$ will cancel out the branch points which are introduced by the factor $\sqrt{s^2 + 1}$. i.e.:

$$L[Au_i'(x', 0)] = \phi(s)/\sqrt{s^2 + 1} \tag{10}$$

where $\phi(s)$ is a single valued analytic function for $s < k$ except for a finite number of poles. An expansion satisfying this requirement is:

$$Au_t'(x', 0) = \sum_{n=0}^{\infty} [b_n x'^n J_n(x') + c_n x'^{n+1} J_n(x')] + \sum_{m=0}^{M} d_m \delta(x' - t_m') \quad (11)$$

where $\delta(x' - t_m')$ represents the Dirac delta function. The last term is included to provide for a finite number of ordinary discontinuities of u at the points $x' = t_m'$. The coefficients d_m are determined by the magnitude of these discontinuities. Considering only the continuous part of $Au_t'(x', 0)$, the coefficients b_n and c_n may be obtained by expanding $J_n(x')$ in a power series and comparing the result with terms of like powers of x' in a Maclaurin expansion of $Au_t'(x', 0)$. A term by term comparison of the continuous portion of Eq. (11) with its Neumann series expansion[4] shows that a sufficient condition for the convergence of Eq. (11) is $0 \leqq x' \leqq 1$. Hence, for $0 \leqq (B/A) \leqq 1$ convergence is assured in the region of main interest, $0 \leqq x \leqq 1$. The convergence of Eq. (11) in the region $x' > 1$ has not, as yet, been investigated.

Evaluation of $g_1(\xi')$ may now be performed quite readily with the aid of standard tables of Laplace transforms. Substituting Eq. (11) into Eq. (8), we have

$$L[g_1(\xi')] = \sum_{n=0}^{\infty} \left[\frac{2^n \Gamma(n + \frac{1}{2})}{\sqrt{\pi}} \cdot \frac{b_n s}{(s^2 + 1)^n} + \frac{2^{n+1} \Gamma(n + \frac{3}{2})}{\sqrt{\pi}} \cdot \frac{c_n s}{(s^2 + 1)^{n+1}} \right] + \sum_{m=0}^{M} d_m s \sqrt{s^2 + 1}\, e^{-st_m} \quad (12)$$

Upon inversion and after replacing ξ' by $(B/A)\xi$, the result is

$$\begin{aligned} g(\xi) = u_{tt}'(\xi, 0) &= b_0 \delta(\xi) \\ &+ \sum_{n=0}^{\infty} \frac{(B/A)\sqrt{2}}{\Gamma(n)} [\Gamma(n + \tfrac{1}{2}) b_n \{(B/A)\xi\}^{n-\frac{1}{2}} J_{n-\frac{1}{2}}\{(B/A)\xi\}] \\ &+ \sum_{n=0}^{\infty} \frac{(B/A)\sqrt{2}}{\Gamma(n)} \left[\frac{\Gamma(n + \frac{3}{2})}{n} c_n \{(B/A)\xi\}^{n+\frac{1}{2}} J_{n+\frac{1}{2}}\{(B/A)\xi\} \right] \\ &+ \sum_{m=0}^{M} d_m \left[\frac{J_1[(B/A)(\xi - t_m)]}{(\xi - t_m)} + \frac{\delta'(\xi - t_m)}{B/A} \right] \end{aligned} \quad (13)$$

The explicit expression for $u_{tt}'(\xi, 0)$ given in Eq. (13) is now substituted into Eq. (5) to obtain the complete determination of $u_t'(x, t)$ in the region $x - At \leqq 1$.

Determination of $u(x, t)$ on the boundary $t = 0$, $0 \leqq x \leqq 1$. If Eq. (5) is multiplied by $[e^{-(b/a)t}$ and integrated with respect to t, an expression is obtained for $u(x, t)$. Using τ as a dummy variable of integration and noting that, from Eq. (2b), $u(x, x/A) = 0$, we obtain

$$\int_t^{x/A} u_\tau(x,\tau)\,d\tau = -u(x,t) = \tfrac{1}{2}\int_t^{x/A} e^{-(b/a)t}\,u'_\tau(x-A\tau,0)\,d\tau$$

$$+\frac{1}{2A}\int_t^{x/A} e^{-(b/a)\tau}\left[\int_{0-}^{x-A\tau} g(\xi)J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\,d\xi\right]d\tau \tag{14}$$

$$-\frac{1}{2A}\int_t^{x/A} e^{-(b/a)\tau}\left[\int_{0-}^{x-A\tau} u'_\tau(\xi,0)\,\frac{d}{d\tau}\{J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\}\,d\xi\right]d\tau$$

The value of $u(x, 0)$ is obtained by placing $t = 0$ in the limits of integration in Eq. (14). By reversing the order of integration in the last two terms of Eq. (14) the result is

$$-u(x,0) = \tfrac{1}{2}\int_0^{x/A} e^{-(b/a)\tau}\,u'_\tau(x-A\tau,0)\,d\tau$$

$$+\frac{1}{2A}\int_{0-}^{x} g(\xi)\left[\int_0^{(x-\xi)/A} e^{-(b/a)\tau}\,J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\,d\tau\right]d\xi \tag{15}$$

$$-\frac{1}{2A}\int_{0-}^{x} u'_\tau(\xi,0)\left[\int_0^{(x-\xi)/A} e^{-(b/a)\tau}\,\frac{d}{d\tau}\{J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\}\,d\tau\right]d\xi$$

Eq. (15) may be further simplified by integrating the last term by parts to give the result:

$$-u(x,0) = \frac{1}{2A}\int_{0-}^{x} u'_\tau(\xi,0)J_0[(B/A)(x-\xi)]\,d\xi$$

$$+\tfrac{1}{2}\int_{0-}^{x}[\{g(\xi)\tau/A\} - (b/aA)u'_\tau(\xi,0)] \tag{16}$$

$$\cdot\left[\int_0^{(x-\xi)/A} e^{-(b/a)\tau}\,J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\,d\tau\right]d\xi$$

Now since the two integrals on the right-hand side of Eq. (16) are both of the form

$$\int_{0-}^{x} F(\xi)I(x-\xi)\,d\xi$$

their evaluation may be carried out with the aid of the Laplace transform and the convolution theorem.

Denoting

$$\int_0^{(x-\xi)/A} e^{-(b/a)\tau}\,J_0[(B/A)\sqrt{(x-\xi)^2-A^2\tau^2}]\,d\tau \quad \text{by} \quad I\left(\frac{x-\xi}{A}\right)$$

the transform of $-u(x, 0)$ is

$$L[-u(x,0)] = \frac{1}{2A}\,\frac{L[u'_\tau(x,0)]}{\sqrt{s^2+B^2/A^2}} + \frac{L[g(x)/A - (b/aA)u'_\tau(x,0)]L[I(x/A)]}{s} \tag{17}$$

where[3]

$$L\left[I\left(\frac{x}{A}\right)\right] = \frac{s/A}{\sqrt{s^2 - B^2/A^2}\,(\sqrt{s^2 + B^2/A^2} + (b/aA)} \tag{18}$$

Eq. (18) and Eq. (8) are now substituted into Eq. (17). After some algebraic manipulation the result can be written as

$$L[-u(x, 0)] = \frac{1}{A^2}\frac{L[g(x) - (b/a)u_r'(x, 0)]}{s^2 + (B^2/A^2) - (b^2/a^2A^2)} \tag{19}$$

Letting $(B^2/A^2) - (b^2/a^2A^2) = k^2$ and taking the inverse transform, we obtain the final result:

$$-u(x, 0) = (1/A^2)\int_{0-}^{x} [g(\xi) - (b/a)u_r'(\xi, 0)]k^{-1} \sin k(x - \xi)\, d\xi \tag{20}$$

For the case $c = 0$, $A = a = 1$ we have $k = 0$ and $B = b$. Eq. (1) then becomes:

$$\partial^2 u/\partial t^2 + 2b\partial u/\partial t = \partial^2 u/\partial x^2 \tag{21}$$

for which the solution on the boundary $t = 0$ is, from Eq. (20)

$$-u(x, 0) = \int_{0-}^{x} [g(\xi) - bu_i'(\xi, 0)](x - \xi)\, d\xi \tag{22}$$

Application to the case of supersonic shear flow. The linearized equation governing the perturbation pressure in a two dimensional supersonic shear flow is[4]

$$(M_1^2 - 1)\frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial y^2} + 2\frac{dM_1/dy}{M_1}\frac{\partial P}{\partial y} = 0 \tag{23}$$

The normal form of Eq. (23) is arrived at by the transformation

$$r = \int_0^y \sqrt{M_1^2 - 1}\, dy \tag{24}$$

which gives

$$\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial r^2} + \frac{(dM_1/dy)[M_1^2 - 2]}{M_1[M_1^2 - 1]^{3/2}} \cdot \frac{\partial P}{\partial r} = 0 \tag{25}$$

In general, a solution of Eq. (25) is difficult to construct because of the complexity of the coefficient of $\partial P/\partial r$. However, in the particular case of a family of Mach number profiles characterized by

$$\frac{(dM_1/dy)[M_1^2 - 2]}{M_1[M_1^2 - 1]^{3/2}} = 2\lambda = \text{Constant} \tag{26}$$

the differential equation reduces to the simple form

$$\partial^2 P/\partial x^2 - \partial^2 P/\partial r^2 + 2\lambda\partial P/\partial r = 0 \tag{27}$$

which is of the form of Eq. (1) with

$$a = 1, \qquad b = -\lambda, \qquad c = 0.$$

The assumption made in Eq. (26) represents a very special case and the behavior of an airfoil in such a flow is not necessarily typical. Nevertheless, this assumption has proven useful in providing a simple solution as well as an intuitive picture of the shear flow phenomenon, and can be used to serve as a check on more general considerations.[5]

Boundary conditions. For an airfoil of thin arbitrary profile, a boundary condition is provided by the requirement of tangential flow on the airfoil surface.[6] With the aid of the y component of the linearized momentum equation and the assumption that the airfoil is sufficiently thin for the boundary condition to be satisfied at $r = 0$, we have

$$(P_r)_{r=0} = -(\gamma M_0^2/\sqrt{M_0^2 - 1})d\alpha(x)/dx \tag{28}$$

where $\alpha(x)$ is the local profile slope, and M_0 is the Mach number at $r = 0$. The remaining boundary condition is that of no incoming disturbance from the region upstream of the airfoil. These boundary conditions are the equivalent of those formulated in Eq. (2a) and Eq. (2b). Since the differential equation for the perturbation pressure P is of the same form as Eq. (21), the solution $P(x, 0)$ is immediately given by Eq. (22).

$$-P(x, 0) = \int_{0_-}^{x} [g(\xi) + \lambda P_r(\xi, 0)](x - \xi)\, d\xi \tag{29}$$

The velocity profile. In evaluating the extent to which this special case may be typical of the general phenomenon, it is useful to study the various shapes of velocity profiles which are possible under the assumption made in Eq. (26). Direct integration of this relation gives the result:

$$2 \tan^{-1}\sqrt{M_1^2 - 1} + 1/\sqrt{M_1^2 - 1} = 2\lambda y + \text{constant} \tag{30}$$

From a plot of Eq. (30), which is given in Figure (2), it is seen that there are two possible branches of M_1 vs y, branch AB for which $1 < M_1 \leqq \sqrt{2}$, and branch BC for which $M_1 \geqq \sqrt{2}$. The desired value of M_1 at $y = 0$ may be obtained by the proper choice of the arbitrary constant. To some extent, the slope of the profile at $y = 0$ may be set by the choice of λ. A set of representative profiles is illustrated in Figure (3).

In the case of the profiles shown in Figure (3b) and Figure (3c), Eq. (30) yields no real values for M_1 above a certain point whose location depends on the choice of λ and the arbitrary constant. In these cases the profiles are modified by assigning an arbitrary supersonic profile above the point $y = y_D$. Alternatively, a second boundary or wall may be placed above this point. If y_D is sufficiently large so that waves reflected from this level do not impinge upon the airfoil surface, it is obvious that the pressure on the airfoil is independent of the choice of the profile above $y = y_D$ as long as it is supersonic. The point

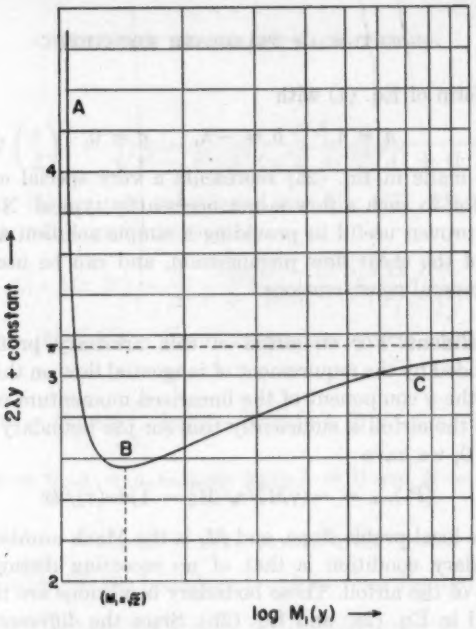

FIG. 2. The main stream profile for the case
$2 \tan^{-1}\sqrt{M_1^2 - 1} + 1/\sqrt{M_1^2 - 1} = 2\lambda y + \text{constant}$

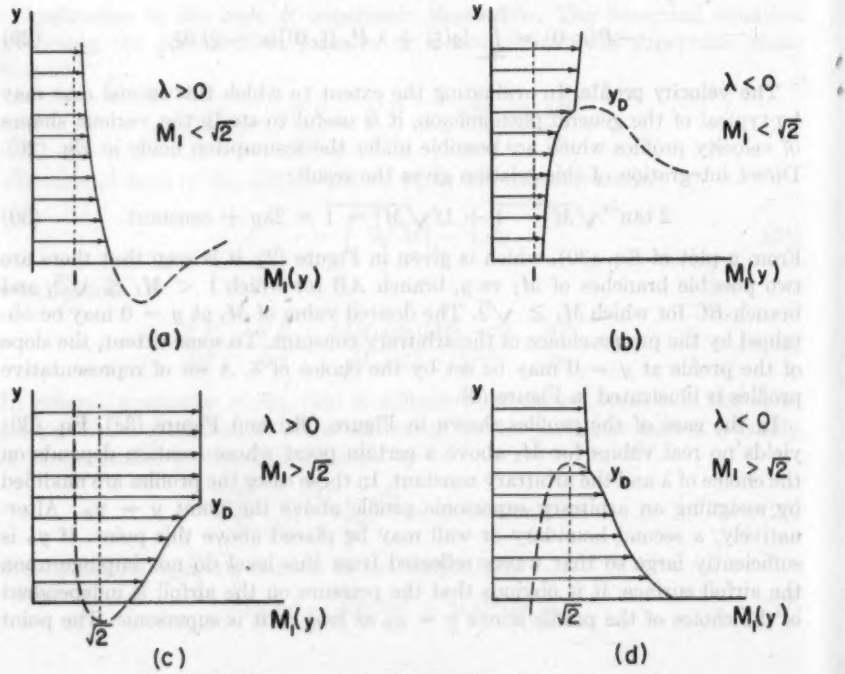

FIG. 3. Profiles represented by Equation 30 (not to scale)

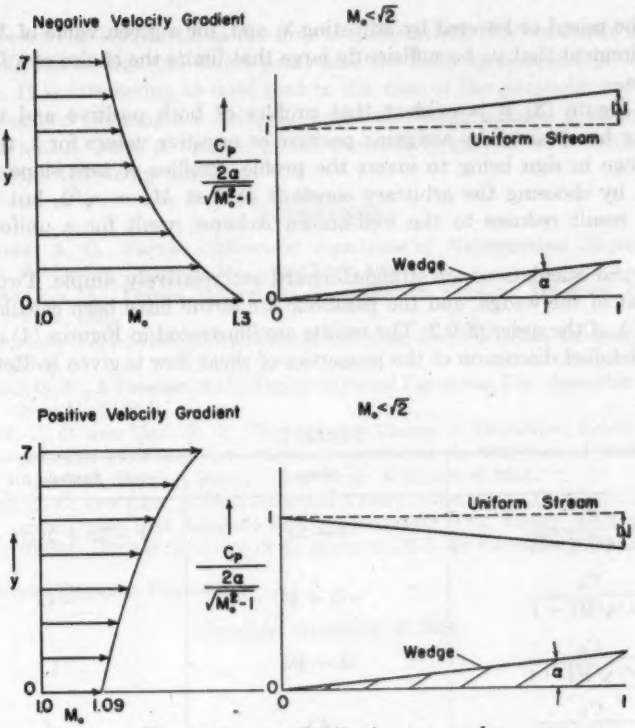

FIG. 4. Pressure distribution on a wedge

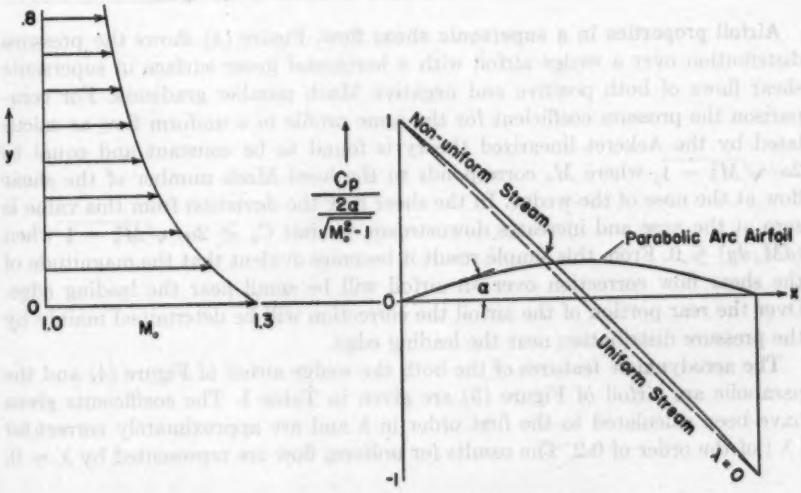

FIG. 5. Pressure distribution on a parabolic arc airfoil ($\lambda = 0.2$)

y_D may be raised or lowered by adjusting λ, and, for a given value of M_0, it is the requirement that y_D be sufficiently large that limits the choice of λ for these two cases.

From Figure (3) it is evident that profiles of both positive and negative slope may be obtained by assigning positive or negative values for λ, the effect of a change in sign being to invert the profile. Profiles of zero slope may be obtained by choosing the arbitrary constant so that $M_0 = \sqrt{2}$, but for this case the result reduces to the well-known Ackeret result for a uniform free stream.

Numerical calculations are straightforward and relatively simple. Two simple cases, that of the wedge, and the parabolic arc airfoil have been calculated for values of λ of the order of 0.2. The results are illustrated in Figures (4) and (5). A more detailed discussion of the properties of shear flow is given in References 5 and 6.

TABLE I

	Wedge	Parabolic Arc
$\frac{C_p}{2\alpha/\sqrt{M_0^2-1}}$	$1+\lambda x$	$1-2x+2\lambda x(1-x)$
$\frac{C_L}{2\alpha/\sqrt{M_0^2-1}}$	$-(1+\frac{1}{2}\lambda)$	$-\frac{1}{2}\lambda$
$\frac{C_D}{2\alpha^2/\sqrt{M_0^2-1}}$	$(1+\frac{1}{2}\lambda)$	$\frac{1}{2}$
$\frac{C_M}{2\alpha/\sqrt{M_0^2-1}}$	$\frac{1}{12}\lambda$	$-\frac{1}{6}$

Airfoil properties in a supersonic shear flow. Figure (4) shows the pressure distribution over a wedge airfoil with a horizontal lower surface in supersonic shear flows of both positive and negative Mach number gradients. For comparison the pressure coefficient for the same profile in a uniform flow as calculated by the Ackeret linearized theory is found to be constant and equal to $2\alpha/\sqrt{M_0^2-1}$, where M_0 corresponds to the local Mach number of the shear flow at the nose of the wedge. In the shear flow the deviation from this value is zero at the nose and increases downstream, so that $C_p \gtrless 2\alpha/\sqrt{M_0^2-1}$ when $(dM/dy) \lesseqgtr 0$. From this simple result it becomes evident that the magnitude of the shear flow correction over an airfoil will be small near the leading edge. Over the rear portion of the airfoil the correction will be determined mainly by the pressure distribution near the leading edge.

The aerodynamic features of the both the wedge airfoil of Figure (4) and the parabolic arc airfoil of Figure (5) are given in Table I. The coefficients given have been calculated to the first order in λ and are approximately correct for $|\lambda|$ of the order of 0.2. The results for uniform flow are represented by $\lambda = 0$.

In each case the lower half of the flow is undisturbed. The Moment coefficient C_M is referred to the mid-chord point and the nose up moment is considered positive. It is interesting to note that in the case of the parabolic arc both the wave drag and pitching are independent of the Mach number gradients to the first order in λ. This, of course, is not the case in general as is shown by the properties of the wedge airfoil.

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NOTE ON THE DISTRIBUTION OF SINGULAR DIRECTIONS OF A TWO-DIMENSIONAL SHOCK

By S. F. Shen

1. Introduction. To construct the flow over a pointed two-dimensional obstacle in supersonic stream, it is at least plausible to assume that an analytic body shape starting from the vertex will lead to an attached analytic shock shape. Indeed, Thomas (1) has shown that the relation between the curvatures of the shock and the body is representable in the form:

$$d^nK/d\eta^n = G_n(M, \alpha)(d^nk/ds^n) + H_n \tag{1}$$

where K = curvature of the body at the vertex
k = curvature of the shock at the vertex
η = arc length along the body
s = arc length along the shock
M = Mach number of the incoming supersonic stream
α = angle between the incoming stream and the shock

and H_n is a function involving the lower order derivatives. Hence in general a formal power series solution can be obtained. Thomas further defined the angle α to be singular at given M if any of the G_n's vanish. Thomas then showed (2) that the distribution of the singular directions is dense in the interval $\beta(M) \leqq \alpha \leqq 90°$, β being the minimum value of α giving subsonic flow after the shock. Consequently, if the initial vertex angle of the body leads to subsonic flow behind the shock, the power series approach based on eq. (1) will fail.

Thomas has actually computed (1) the functions G_0 and G_1 and found the numerical values of the singular directions α_0 (for $G_0 = 0$) and α_1 (for $G_1 = 0$), for various Mach numbers. The quantity α_1 turns out to be double-valued. One may recall that the singular direction α_0 (for $G_0 = 0$) is the well-known "Crocco point" first discovered by Crocco (3), the significance of which has been discussed by Guderley (4). Allowing for a singularity at the vertex in the hodograph plane, Guderley showed that the shock would start with an infinite curvature when the initial vertex angle exceeds the "Crocco point." There would be neither reversed curvature as dictated by such relation as eq. (1), nor a detached shock as conjectured by Crocco.

It is not clear, however, what kind of significance one should attach to α_1 (for $G_1 = 0$), as well as the "higher order" singular directions α_m (for $G_m = 0$), $m > 1$. Although the reverse of higher order curvatures d^mk/ds^m does not seem to be as difficult to imagine as the reverse curvature itself, the formal conclusion that d^mk/ds^m changes sign as α passes through α_m does not immediately follow, since after α_m the assumed analytic relation may not hold. One can only say that complications arise. When the singular directions for $G_m = 0$ are multi-valued, such as the two roots α_1, it seems logical to choose the lowest value as the representative one. The m^{th} order derivative always exists for all α's less than this value, but not necessarily so above it. Henceforth, α_m will be used to denote

the lowest root of $G_m = 0$ at given M. The set $\{\alpha_m\}$ is discrete and denumerable in the interval $\beta \leqq \alpha \leqq 90°$. It is felt that some insight to the manner in which the analyticity of the shock breaks down might be gained by studying the distribution of this set $\{\alpha_m\}$. The viewpoint here is that α_m might be the limit of the existence of the m^{th} order derivative, as a generalization of the "Crocco point" phenomenon.

It is shown in this note that these singular directions α_m are actually ordered. That is,

$$\alpha_d > \alpha_0 > \alpha_1 > \cdots > \alpha_{n-1} > \alpha_n > \cdots > \beta$$

and $\alpha_n \to \beta$ as $n \to \infty$. Here α_d is the shock angle giving maximum deflection to the incoming stream. The shocks in the interval $\alpha_d < \alpha \leqq 90°$ are the so-called "strong family" for a given vertex angle of the body and are not observed under ordinary conditions.

After the α_m's have been ordered as above, one might interpret the physical meaning as follows: For a given M, imagine the vertex angle of a curved body to be gradually increased until subsonic flow begins to prevail behind the shock. Further increase of the vertex angle destroys the analyticity of the shock shape. The highest order derivatives may run into trouble first. The lower order ones follows the higher ones in order. This result was, in fact, first surmised by Professor C. C. Lin during a discussion with the author, which inspired the investigation. The author also acknowledges his many helpful criticisms.

2. The zeros of $G_n(M, \alpha)$ for given M. The singular directions are determined from the condition $G_n = 0$. This condition was shown (cf. eqs. (53) through (56) of ref. 1) to be equivalent to

$$\sin [(n + 1)\theta + \psi] = 0 \tag{2}$$

In equation (2), the notation is that used by Thomas, namely

$$\begin{aligned}
\bar{\theta} &= \tan^{-1}[\sqrt{1 - m^2} \tan (\alpha - \omega)],\ 0 \leqq \bar{\theta} \leqq \tfrac{1}{2}\pi, \\
m &= \text{Mach number immediately after the shock at the vertex,} \\
\omega &= \text{initial vertex angle of the body} \\
\psi &= \tan^{-1}(-a/-b),\ \tfrac{1}{2}\pi \leqq \psi \leqq \tfrac{1}{2}\pi \text{ for } \beta \leqq \alpha \leqq \tfrac{1}{2}\pi; \\
a &= [4/(\gamma + 1) - (\rho - \rho_1)/\rho] \cos^2 \theta - [(\rho - \rho_1)/\rho_1] \sin^2 \theta, \\
b &= [2/(\gamma + 1)] \sqrt{1 - m^2} \sin 2\theta, \qquad \theta = \alpha - \omega
\end{aligned}$$

and ρ and ρ_1 are the density immediately after and before the shock at the vertex, respectively. Thomas showed that in the neighborhood of each given α in the interval $\beta \leqq \alpha \leqq 90°$, there exists a value of α and a value of n satisfying equation (2), and hence the distribution of the singular directions is dense in this interval.

Now suppose the value of n is given. The singular direction α_n must satisfy

$$\bar{\theta}(M_1, \alpha_n) = k\pi/(n + 1) - \psi(M_1, \alpha_n)/(n + 1) \qquad k = 0, 1, 2, \cdots \tag{3}$$

Obviously the solution α_n is in general multivalued depending on the choice of k, as long as the right-hand side is made to lie between 0 and $\pi/2$. For the limiting

case of $n \to \infty$, the singular direction as solved from the condition $\bar{\theta} = 0$ is the one yielding $m = 1$, i.e., $\alpha = \beta$. The other singular directions for large n are thus close to the point $\alpha = \beta$. It is easily seen that $\psi(M_1, \beta) = -\pi/2$ by definition. In looking for the lowest value of α_n for any given $n \gg 1$, the value of k chosen should therefore be zero. Both $\bar{\theta}$ and ψ being smooth functions of α, one may write down for the *lowest* singular direction of the n^{th} order derivative as

$$\alpha_n \doteq \beta + \frac{1}{(d\bar{\theta}/d\alpha)_{\alpha=\beta}} \frac{\pi}{2(n+1)} \tag{4}$$

when $n \gg 1$. In §3 it is shown that $d\bar{\theta}/d\alpha > 0$. Hence the lowest singular direction for a larger n is always located nearer to the point $\alpha = \beta$ than that for a smaller n, provided $n \gg 1$.

Consider next the general case of arbitrary n. Applying equation (3) to the lowest singular directions for $n = m$ and $n = m + 1$ one gets

$$\bar{\theta}(M_1, \alpha_{m+1}) - \bar{\theta}(M_1, \alpha_m) = +\left[\frac{\psi(M_1, \alpha_m)}{m+1} - \frac{\psi(M_1, \alpha_{m+1})}{m+2}\right] + \left[\frac{k_{m+1}}{m+2} - \frac{k_m}{m+1}\right]\pi \tag{5}$$

where k_{m+1} and k_m stand for the proper values of k for $n = m + 1$ and $n = m$ respectively. Assume now that in the interval $\beta \leqq \alpha \leqq \alpha_d$:

(i) $\bar{\theta}$ is a monotonic increasing function of α;
(ii) ψ is a monotonic increasing function of α;
(iii) $\psi < 0$;

then one may put

$$\bar{\theta}(M_1, \alpha_{m+1}) - \bar{\theta}(M_1, \alpha_m) = (\Delta\theta/\Delta\alpha)_m (\alpha_{m-1} - \alpha_m),$$

$$\frac{\psi(M_1, \alpha_{m+1})}{m+2} - \frac{\psi(M_1, \alpha_m)}{m+1} = -\frac{\psi(M_1, \alpha_m)}{(m+1)(m+2)} + \left(\frac{\Delta\psi}{\Delta\alpha}\right)_m \left(\frac{\alpha_{m+1} - \alpha_m}{m+2}\right)$$

and $k_{m+1} = k_m = 0$ as ψ now lies between $-\pi/2$ and 0, where $\Delta\theta/\Delta\alpha$ and $(\Delta\psi/\Delta\alpha)_m$ are respectively the secant slopes of the $\bar{\theta}$ vs. α and ψ vs. α curves between points α_m and α_{m+1}, and both are positive in the interval $\beta \leqq \alpha \leqq \alpha_d$. By rewriting equation (5), there follows,

$$\alpha_{m+1} - \alpha_m = +\frac{\psi(M_1, \alpha_m)}{(m+1)(m+2)} \Big/ \left\{\left(\frac{\Delta\theta}{\Delta\alpha}\right)_m + \frac{1}{m+2}\left(\frac{\Delta\psi}{\Delta\alpha}\right)_m\right\} < 0. \tag{6}$$

This shows that the *lowest singular direction for a larger n always lies ahead of that for a smaller n in the interval* $\beta \leqq \alpha \leqq \alpha_d$. The validity of the assumptions (i), (ii), and (iii) will be established in the following sections.

3. The function $\bar{\theta}$. By definition

$$\tan\bar{\theta} = \sqrt{1 - m^2}\tan\theta > 0 \text{ for } \beta < \alpha \leqq \alpha_d \text{ ; } = 0 \text{ for } \alpha = \beta$$

The assumption (i) of a monotonically increasing $\bar{\theta}$ is nothing but that of a positive $d\bar{\theta}/d\alpha$ for $\beta \leqq \alpha \leqq \alpha_d$. Differentiating,

$$\sec^2 \tilde{\theta} \frac{d\tilde{\theta}}{da} = \sqrt{1 - m^2} \frac{d \tan \theta}{d\alpha} + \frac{\tan \theta}{2\sqrt{1 - m^2}} \frac{d}{d\alpha}(1 - m^2) \tag{7}$$

It is obvious by examining the conventional plottings of oblique shock properties that in the interval $\beta \leqq \alpha \leqq \alpha_d$,

$$\tan \theta > 0, \qquad d \tan \theta / d\alpha > 0, \qquad d(1 - m^2)/d\alpha > 0$$

Hence equation (7) leads to $d\tilde{\theta}/d\alpha > 0$.

4. The function "a". The function "a" is defined by

$$a = [4/(\gamma + 1) - (\rho - \rho_1)/\rho] \cos^2 \theta - [(\rho - \rho_1)/\rho_1] \sin^2 \theta$$

Thomas showed in Ref. 1 that $a > 0$ for $m > 1$. It is clear that for a normal shock $\alpha = \theta = \pi/2$, "a" becomes negative. At what value of α does "a" change its sign?

By a slight transformation, the value of α where $a = 0$ is found to satisfy

$$(\rho_1/\rho)^2 \tan^2 \alpha + (\rho_1/\rho)(1 - \tan^2 \alpha) + (3 - \gamma)/(\gamma + 1) = 0 \tag{8}$$

Since

$$\rho_1/\rho = (\gamma - 1)/(\gamma + 1) + [2/(\gamma + 1)][1/M_1^2 \sin^2 \alpha],$$

after substituting in equation (8), a quadratic equation in $\sin^2 \alpha$ follows. The solution turns out to be

$$\sin^2 \alpha = \frac{\gamma + 1}{4\gamma}\left[1 - \frac{4}{\gamma + 1}\frac{1}{M_1^2} + \sqrt{1 + 8\frac{\gamma - 1}{\gamma + 1}\frac{1}{M_1^2} + \frac{16}{\gamma + 1}\frac{1}{M_1^4}}\right] \tag{9}$$

One finds by comparison that the right-hand side of equation (9) is exactly the same as that for the solution which gives a maximum deflection of the stream direction. In other words, $a = 0$ *only when* $\alpha = \alpha_d$.

Consider now the variation of "a" from $\alpha = \beta$ to $\alpha = \alpha_d$:

$$\frac{\partial a}{\partial \alpha} = \left[\frac{2(\gamma - 1)}{\gamma + 1} - \left(\frac{\rho_1}{\rho} + \frac{\rho}{\rho_1}\right)\right] \sin 2\theta \frac{\partial \theta}{\partial \alpha} + \left[\cos^2 \theta + \left(\frac{\rho_1}{\rho}\right)^2 \sin^2 \theta\right] \frac{\partial}{\partial \alpha}\left(\frac{\rho}{\rho_1}\right) \tag{10}$$

Using the expression for ρ_1/ρ previously given,

$$\frac{2(\gamma - 1)}{\gamma + 1} - \left(\frac{\rho_1}{\rho} + \frac{\rho}{\rho_1}\right)$$

$$= -\frac{2}{\gamma + 1}\left(1 + \frac{1}{M_1^2 \sin^2 \alpha}\right) - \frac{2(M_1^2 \sin^2 \alpha - 1)}{(\gamma - 1)M_1^2 \sin^2 \alpha + 2} < 0;$$

also obviously $\partial(\rho/\rho_1)/\partial\alpha > 0$. Hence both items on the right-hand side of equation (10) are negative, and $\partial a/\partial\alpha < 0$ everywhere in the interval $\beta \leqq \alpha \leqq \alpha_d$.

5. The function "b". The function "b" is defined by

$$b = -[2/(\gamma + 1)]\sqrt{1 - m^2} \sin 2\theta.$$

Starting from zero at $\alpha = \beta$ where $m = 1$, "b" reaches zero again when $\alpha = \theta = \pi/2$. For $\beta < \alpha \leqq \alpha_d$, therefore, $b < 0$. It will now be shown that $-b$ increases monotonically with α in the interval $\beta \leqq \alpha \leqq \alpha_d$. Consider again the derivative,

$$\partial(-b)/\partial\alpha = -2b[\cot 2\theta(\partial\theta/\partial\alpha) + \tfrac{1}{2}[1/(1-m^2)]\partial(1-m^2)/\partial\alpha] \tag{11}$$

$\partial(-b)/\partial\alpha$ is seen to be positive if the terms in the bracket add up to be positive. The second term involving $(1 - m^2)$ remains positive for all values of α. The derivative $\partial\theta/\partial\alpha$ equals $1 - \partial a/\partial\alpha$, and is therefore positive, increasing monotonically to unity when $\alpha = \alpha_d$. Only the quantity cot 2θ tends to change the sign of $\partial(-b)/\partial\alpha$ when $\theta > \pi/4$. Referring back to the usual formulae for oblique shock, one has

$$1 - m^2 = \frac{\gamma+1}{\gamma-1}\left\{1 - \frac{(\gamma-1)M_1^2+2}{\gamma^2-1} \cdot \left(\frac{2}{h + [2/(\gamma-1)]} + \frac{\gamma-1}{[2\gamma/(\gamma-1)]\,h - 1}\right)\right\} \tag{12}$$

$$\tan\theta = [2/(\gamma+1)]\,[\tfrac{1}{2}(\gamma-1)\tan\alpha + (2/M_1^2\sin 2\alpha)] \tag{13}$$

where $h = M_1^2 \sin^2\alpha$. From (12) there follows

$$\frac{1}{1-m^2}\frac{\partial}{\partial\alpha}(1-m^2) = 2M_1^2\sin 2\alpha\,\frac{(\gamma-1)M_1^2+2}{\gamma^2-1} \cdot \frac{\dfrac{1}{[h+2/(\gamma-1)]^2} + \dfrac{\gamma}{[(2\gamma/(\gamma-1))h-1]^2}}{1 - \dfrac{(\gamma-1)M_1^2+2}{\gamma^2-1}\left\{\dfrac{2}{h+(2/(\gamma-1))} + \dfrac{\gamma-1}{(2\gamma/(\gamma-1))h-1}\right\}} \tag{14}$$

For a given M_1, then, a larger α leads to a larger value of h, which in turn leads to a smaller value of $[1/(1-m^2)]\partial(1-m^2)/\partial\alpha$, as readily seen from equation (14), noting that $\alpha \geqq \beta > \pi/4$. From equation (13) a larger α also leads to a larger θ; hence a larger *negative* value of cot 2θ whenever $\theta > \pi/4$. Thus it is sufficient to check the sign of $\partial(-b)/\partial\alpha$ at $\alpha = \alpha_d$. Write

$$S(M_1^2) = \cot 2\theta_d + \tfrac{1}{2}\{[1/(1-m^2)]\partial(1-m^2)/\partial\alpha\}_d \tag{15}$$

when subscript "d" denoting the value at $\alpha = \alpha_d$. $S = 0$ will correspond to $\partial(-b)/\partial\alpha = 0$. Since $\sin^2\alpha_d$ is a continuous function of M_1^2, so are h_d and θ_d. Furthermore, since $h_d > 1 > (\gamma-1)/2\gamma$, the function $(1-m^2)_d$ cannot have any discontinuity for all $M_1^2 > 1$. One may therefore be sure that $S(M_1^2)$ is a continuous function of M_1^2. The function $S(M_1^2)$ has been computed for $1.05 \leqq M_1^2 \leqq 4.0$, with $\gamma = 1.4$, and results are tabulated in Table 1 and plotted as Fig. 1. There is no need to go beyond $M_1^2 = 4$ since θ_d can easily be shown to be decreasing with M_1^2 and is already found to be less than $\pi/4$ at $M_1^2 = 4$. Both terms on the right-hand side of equation (15) are positive, and therefore $S > 0$ for $M_1^2 > 4$.

For smaller values of M_1^2, however, it is known that $\alpha_d \doteq \pi/2$ and an investigation seems to be in order, although one might be convinced that by extrapolating from Fig. 1, no sudden change of sign is likely to occur.

Putting $M_1^2 = 1 + \mu$, $\mu \ll 1$, one has the following result after some manipulation:

$$b = -[4\mu/(\gamma + 1)][1 - \xi + \cdots]\sqrt{\xi(1 - \xi/2)} \tag{16}$$

where $\xi = \alpha - \beta/\sqrt{\mu/2}$. Hence by differentiation,

TABLE I

The function $S(M_1^2)$

M_1^2	$\sin^2 \alpha_d$	k_d	$\tan \delta_d$	S
1.05	.98425	1.0335	7.692	4.097
1.10	.97010	1.0671	5.398	3.122
1.15	.95746	1.1011	4.381	2.730
1.25	.93569	1.1696	3.353	2.392
1.40	.90987	1.2740	2.608	2.262
1.60	.88494	1.4159	2.094	2.216
2.00	.85389	1.7078	1.583	2.451
3.00	.82401	2.4720	1.091	3.303
4.00	.81695	3.2678	.891	4.307

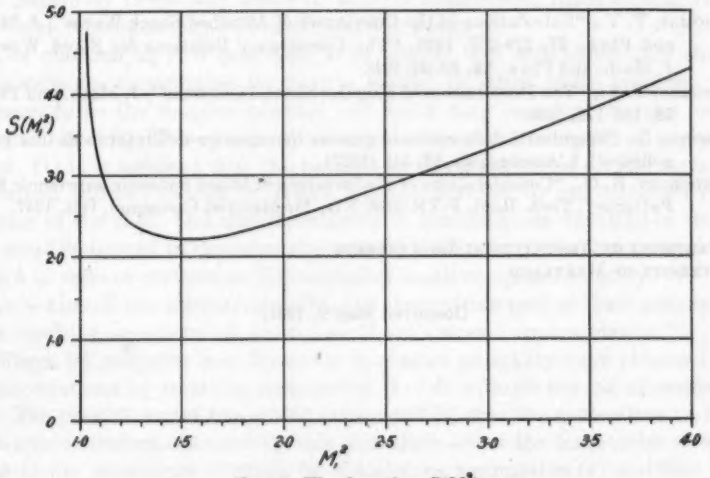

FIG. 1. The function $S(M_1^2)$

$$\frac{\partial(-b)}{\partial\alpha} = \frac{4}{\gamma + 1}\frac{\sqrt{\mu/2}}{\sqrt{\xi(1 - \xi/2)}}[2(1 - \xi)^2 - 1 + \cdots]. \tag{17}$$

Thus $\partial(-b)/\partial\alpha > 0$ if $\xi < 1 - 1/\sqrt{2} = 0.293$. Now the largest value of ξ is at $\alpha = \alpha_d$ or

$$\xi_{\max} = \frac{\alpha_d - \beta}{\sqrt{\mu/2}} = 1 - \sqrt{\tfrac{4}{5}} < 0.293.$$

The slope $\partial(-b)/\partial\alpha$ therefore is still always positive for M_1 very close to unity. Further, since at $\alpha = \alpha_d$

$$\partial(-b)/\partial\alpha = -2bS,$$

one has $S(M_1^2) \sim O(1/\sqrt{\mu})$, tending to $+\infty$ as $\mu \to 0$. This tendency is clearly indicated in Fig. 1.

6. The function "ψ". Having shown the monotonic properties of both "a" and "b", one can finally conclude on ψ defined by $\tan\psi = (-a)/(-b)$ and at $\alpha = \beta$, $\psi = -\pi/2$. Taking the derivative,

$$\sec^2\psi\partial\psi/\partial\alpha = (1/-b)[-(\partial a/\partial\alpha) - \tan\psi\{\partial(-b)/\partial\alpha)\}] \tag{18}$$

For $\beta \leqq \alpha \leqq \alpha_d$, it was shown in equation (4) that $a > 0$, $\partial a/\partial\alpha < 0$, and in equation (5) that $-b > 0$, $\partial(-b)/\partial\alpha > 0$. Hence ψ is always negative and the right-hand side of equation (18) is always positive in $\beta \leqq \alpha \leqq \alpha_d$. It follows immediately that $\partial\psi/\partial\alpha > 0$ for $\beta \leqq \alpha \leqq \alpha_d$. The assumptions (ii) and (iii) in equation (2) are therefore valid.

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STRESS STRAIN RELATIONS IN THE THEORY OF THIN ELASTIC SHELLS*

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1. Introduction. This paper is concerned with the linear theory of small deformations of thin elastic shells. There is no difficulty so far as the equations of equilibrium in this theory are concerned. The problem to which further discussion is given below consists in the formulation of suitable stress strain relations for the theory.

On the one hand one has stress resultants and couples representing the state of stress in the shell, and on the other hand one has expressions for the strains of the stressed shell depending on the deformation of the middle surface. Depending on what assumptions are made these stress strain relations may appear in a variety of forms, differing from each other, presumably, only through terms which are small compared to the principal terms which are common to all forms.

A considerable literature exists on the subject. We may refer in particular to two papers by Green and Zerna [1] and by Hildebrand, Reissner and Thomas [2] where references may be found to earlier work.

The question as it is seen here is as follows. For sufficiently thin shells of homogeneous construction we have a system of stress strain relations which appears to be the simplest possible and which may be called "Love's first approximation" [2]. This system of relations is based on two types of approximations; (1) it is assumed that the normals to the undeformed middle surface are deformed without change of length into the normals to the deformed middle surface of the shell; and that the stresses in the thickness direction of the shell are small compared to the other stresses; (2) it is assumed that the ratio of thickness h to radii of curvatures R is negligibly small compared to unity.

Love himself has attempted to free the theory from part of these assumptions. The resulting equations are known as "Love's second approximation."

Flügge [3] and after him Byrne [4] in greater generality have obtained stress strain relations by retaining assumption (1) but without the use of assumption (2). The present writer has earlier contended [5] that the corrections to Love's first approximation obtained by this procedure are of the same order of magnitude as the corrections obtained by abandoning assumption (1) and that therefore if corrections to Love's first approximation were desired these should be obtained by *simultaneously* abandoning assumptions (1) and (2). This has been done for homogeneous shells in [2] and also for a class of non-homogeneous shells, generally referred to as sandwich-type shells [6].

In what follows we reconsider the problem in a manner which represents some simplification compared with our earlier work [2] both in regard to the form of the results and in regard to the method of derivation. The latter is accomplished by

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using a variational principle which was recently formulated [7] in the expectation that it would simplify work of the nature here carried out.

In order to fix the ideas the following derivations are restricted to the case of axi-symmetrical deformations of shells of revolution. It will be apparent however that this is not a necessary restriction.

Furthermore, in order to see from what sources the various terms which represent the deviations from Love's first approximation originate we assume as in [2] a medium for which the directions of the normals to the middle surface of the shell are preferred directions in regard to the elastic behaviour of the shell.

2. Stresses and strains in surface-of-revolution coordinates. The equation of the middle surface of the shell is written in the form

$$r = r(\xi), \qquad z = z(\xi) \tag{1}$$

so that ξ, together with the polar angle θ in the x, y-plane, are curvilinear coordinates on the middle surface of the shell. The third coordinate in the surface-of-revolution system is the distance ζ from the middle surface, measured along the normals to the middle surface.

The linear element for this system of coordinates is of the form

$$ds^2 = \alpha^2\left(1 + \frac{\zeta}{R_\xi}\right)^2 d\xi^2 + r^2\left(1 + \frac{\zeta}{R_\theta}\right)^2 d\theta^2 + d\zeta^2 \tag{2}$$

where

$$\alpha^2 = (r')^2 + (z')^2 \tag{3}$$

and

$$\frac{1}{R_\xi} = -\frac{\phi'}{\alpha}, \qquad \frac{1}{R_\theta} = -\frac{\sin\phi}{r}, \tag{4}$$

ϕ being the sloping angle of the tangents to a meridian curve of the middle surface. Primes indicate differentiation with respect to ξ.

Normal stresses are designated by σ_ξ, σ_θ and σ_ζ. Since axially symmetric deformations with no circumferential displacement are assumed the only component of shearing stress which needs to be considered is the transverse shearing stress on sections $\xi = \text{const.}$ which will be designated by τ_ξ.

We designate by U and W the components of displacement tangential to the meridian curves and in the directions of the normals to the middle surface, these being the only components of displacements which occur. The quantities U and W, as well as σ_ξ, σ_θ, σ_ζ and τ_ξ are assumed to be independent of the coordinate θ.

For what follows we do not need to formulate the equations of equilibrium for the components of stress but we do need the appropriate expressions for components of strain. These are

$$\varepsilon_\xi = \frac{(\partial U/\partial\xi) + (\alpha W/R_\xi)}{\alpha(1 + \zeta/R_\xi)}$$

$$\varepsilon_\theta = \frac{(r' U/\alpha) + (rW/R_\theta)}{r(1 + \zeta/R_\theta)} \tag{5}$$

$$\varepsilon_\zeta = \partial W/\partial\zeta$$

$$\gamma_\zeta = \frac{\partial W/\partial\xi}{\alpha(1 + \zeta/R_\xi)} + \left(1 + \frac{\zeta}{R_\xi}\right)\frac{\partial}{\partial\zeta}\left(\frac{U}{1 + \zeta/R_\xi}\right).$$

Stress strain relations are taken in the form

$$\varepsilon_\xi = \frac{\sigma_\xi - \nu\sigma_\theta}{E} - \frac{\nu_\zeta\sigma_\zeta}{E_\zeta}, \qquad \varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_\xi}{E} - \frac{\nu_\zeta\sigma_\zeta}{E_\zeta}$$

$$\varepsilon_\zeta = \frac{\sigma_\zeta - \nu_\zeta(\sigma_\xi + \sigma_\theta)}{E_\zeta}, \qquad \gamma_\zeta = \frac{\tau_\zeta}{G_\zeta} \tag{6}$$

where E, ν, E_ζ, ν_ζ and G_ζ are independent constants.

3. Approximations for stresses and displacements. We write

$$\sigma_\xi = \frac{\dfrac{N_\xi}{h} + \dfrac{M_\xi}{h^2/6}\dfrac{\zeta}{h/2}}{(1 + \zeta/R_\theta)}, \qquad \sigma_\theta = \frac{\dfrac{N_\theta}{h} + \dfrac{M_\theta}{h^2/6}\dfrac{\zeta}{h/2}}{(1 + \zeta/R_\xi)}$$

$$\sigma_\zeta = \frac{\dfrac{S}{2h/3}\left[1 - \left(\dfrac{\zeta}{h/2}\right)^2\right]}{(1 + \zeta/R_\xi)(1 + \zeta/R_\theta)}, \qquad \tau_\zeta = \frac{\dfrac{Q}{2h/3}\left[1 - \left(\dfrac{\zeta}{h/2}\right)^2\right]}{(1 + \zeta/R_\theta)}. \tag{7}$$

The quantities N_ξ, N_θ and Q are the conventional stress resultants, M_ξ and M_θ the conventional stress couples, and S is the average transverse normal stress. There are two free quantities in the primary stresses σ_ξ and σ_θ and only one each in what are expected to be secondary stresses. The denominators in (7) are introduced in order that the above designations as resultants, couples and average are exactly true for all values of R_ξ and R_θ. The quantity h represents the wall thickness of the shell and the above choice of σ_ζ and τ_ζ means that no loads are applied to the surfaces $\zeta = \pm h/2$ of the shell.

The assumptions concerning σ_ξ, σ_θ and τ_ζ are self-explanatory. Regarding σ_ζ, we take into account that the transverse normal stress is primarily due to the couples M_ξ and M_θ for reasons of statics and as such is nearly symmetrical about the middle surface.

Approximations for the components of displacement which are qualitatively consistent with the above approximations for the stresses are as follows

$$U = U_M + \zeta\beta, \qquad W = W_M + \zeta\varepsilon_{\zeta M}. \tag{8}$$

In (8) U_M and W_M are the components of displacement of point of the middle surface, β is the change of slope of the normal to the middle surface and $\varepsilon_{\zeta M}$ is

the transverse normal strain at the middle surface. All four quantities are here functions of the coordinate ξ only.

Introduction of (8) into the components of strain (5) leads to the following expressions

$$\varepsilon_\xi = \frac{\varepsilon_{\xi M} + \zeta\kappa_\xi^*}{1 + \zeta/R_\xi}, \qquad \varepsilon_\theta = \frac{\varepsilon_{\theta M} + \zeta\kappa_\theta^*}{1 + \zeta/R_\theta}$$
$$\varepsilon_\zeta = \varepsilon_{\zeta M}, \qquad \gamma_\zeta = \frac{\gamma_{\zeta M} + \kappa_\zeta}{1 + \zeta/R_\xi} \tag{9}$$

where

$$\varepsilon_{\xi M} = \frac{U_M'}{\alpha} + \frac{W_M}{R_\xi}, \qquad \varepsilon_{\theta M} = \frac{r'U_M}{\alpha r} + \frac{W_M}{R_\theta}$$
$$\kappa_\xi^* = \kappa_\xi + \frac{\varepsilon_{\zeta M}}{R_\xi}, \qquad \kappa_\theta^* = \kappa_\theta + \frac{\varepsilon_{\zeta M}}{R_\theta}$$
$$\kappa_\xi = \frac{\beta'}{\alpha}, \qquad \kappa_\theta = \frac{r'\beta}{\alpha r} \tag{10}$$
$$\gamma_{\zeta M} = \frac{W_M'}{\alpha} - \frac{U_M}{R_\xi} + \beta, \qquad \kappa_\zeta = \frac{\varepsilon_{\zeta M}'}{\alpha}.$$

Notice should be taken of the difference between κ_ξ and κ_θ which represent the changes of curvature of the middle surface of the shell and the quantities κ_ξ^* and κ_θ^* which include a term due to the transverse normal strain in the shell.

The problem now is to obtain a system of relations among the stress resultants and couples on the one hand and the strain quantities $\varepsilon_{\xi M}$, $\varepsilon_{\theta M}$, κ_ξ, κ_θ and $\gamma_{\zeta M}$ on the other hand.

4. Derivation of approximate stress strain relations. In order to obtain approximate stress strain relations we use a variational equation the Euler equations of which are both differential equations of equilibrium and stress strain relations [7].

While for the present problem nearly all equilibrium equations for stress resultants and couples are known equations we shall see that application of the variational equation furnishes an extra equilibrium equation which is in fact needed for a complete system of differential equations.

The appropriate form of the variational equation is here

$$\delta \iiint \Bigg\{ \sigma_\xi \varepsilon_\xi + \sigma_\theta \varepsilon_\theta + \sigma_\zeta \varepsilon_\zeta + \tau_\zeta \gamma_\zeta - \frac{1}{2}\left[\frac{\sigma_\xi^2 + \sigma_\theta^2 - 2\nu\sigma_\xi\sigma_\theta}{E} + \frac{\sigma_\zeta^2 - 2\nu_\zeta\sigma_\zeta(\sigma_\xi + \sigma_\theta)}{E_\zeta} + \frac{\tau_\zeta^2}{G_\zeta} \right] \Bigg\} \cdot \left(1 + \frac{\zeta}{R_\xi}\right)\left(1 + \frac{\zeta}{R_\theta}\right) r\alpha \, d\xi \, d\theta \, d\zeta = 0. \tag{11}$$

In this we introduce stresses and strains from (7) and (9), which leads to the following variational equation

$$\delta \iiint \left\{ \left(\frac{N_\xi}{h} + \frac{M_\xi}{h^2/6}\frac{\zeta}{h/2} \right)(\varepsilon_{\xi M} + \zeta \kappa_\xi^*) + \left(\frac{N_\theta}{h} + \frac{M_\theta}{h^2/6}\frac{\zeta}{h/2} \right)(\varepsilon_{\theta M} + \zeta \kappa_\theta^*) \right.$$
$$+ \frac{S}{2h/3}\left[1 - \left(\frac{\zeta}{h/2} \right)^2 \right] \varepsilon_{\xi M} + \frac{Q}{2h/3}\left[1 - \left(\frac{\zeta}{h/2} \right)^2 \right](\gamma_{\xi M} + \zeta \kappa_\xi)$$
$$- \frac{1}{2E}\left[\frac{1 + \zeta/R_\xi}{1 + \zeta/R_\theta}\left(\frac{N_\xi}{h} + \frac{M_\xi}{h^2/6}\frac{\zeta}{h/2} \right)^2 + \frac{1 + \zeta/R_\theta}{1 + \zeta/R_\xi}\left(\frac{N_\theta}{h} + \frac{M_\theta}{h^2/6}\frac{\zeta}{h/2} \right)^2 \right.$$
$$\left. - 2\nu \left(\frac{N_\xi}{h} + \frac{M_\xi}{h^2/2}\frac{\zeta}{h/2} \right)\left(\frac{N_\theta}{h} + \frac{M_\theta}{h^2/6}\frac{\zeta}{h/2} \right) \right] \tag{12}$$
$$- \frac{1}{2E_\zeta}\left[\frac{\left(\frac{S}{2h/3} \right)^2 \left[1 - \left(\frac{\zeta}{h/2} \right)^2 \right]^2}{(1 + \zeta/R_\xi)(1 + \zeta/R_\theta)} \right.$$
$$\left. - 2\nu_\zeta \left(\frac{S}{2h/3} \right)\left[1 - \left(\frac{\zeta}{h/2} \right)^2 \right] \frac{\frac{N_\xi}{h} + \frac{M_\xi}{h^2/6}\frac{\zeta}{h/2}}{1 + \zeta/R_\theta} + \frac{\frac{N_\theta}{h} + \frac{M_\theta}{h^2/6}\frac{\zeta}{h/2}}{1 + \zeta/R_\xi} \right]$$
$$\left. - \frac{1}{2G_\zeta}\frac{1 + \zeta/R_\xi}{1 + \zeta/R_\theta}\left(\frac{Q}{2h/3} \right)^2 \left[1 - \left(\frac{\zeta}{h/2} \right)^2 \right]^2 \right\} \alpha r \, d\xi \, d\theta \, d\zeta = 0.$$

The integrations with respect to ζ may now be carried out. At this point we take account of the fact that h/R is supposed to be small compared with unity by retaining only the leading terms in (h/R) in the development. We write

$$\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1} \, d\zeta \sim h\left[1 + \frac{1}{12}\left(\frac{h}{R_1} - \frac{h}{R_2} \right)\frac{h}{R_1} \right]$$
$$\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1} \zeta \, d\zeta \sim -\frac{h^2}{12}\left(\frac{h}{R_1} - \frac{h}{R_2} \right) \tag{13}$$
$$\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1} \zeta^2 \, d\zeta \sim \frac{h^3}{12}\left[1 + \frac{3}{20}\left(\frac{h}{R_1} - \frac{h}{R_2} \right)\frac{h}{R_1} \right]$$

Since the effect of the stress quantities S and Q will be by itself small compared to the effect of the quantities N and M we may for those terms which involve S and Q disregard h/R entirely in comparison with unity.

Equation (12) now assumes the form

$$\delta \iint \left\{ N_\xi \varepsilon_{\xi M} + N_\theta \varepsilon_{\theta M} + M_\xi \kappa_\xi + M_\theta \kappa_\theta + Q\gamma_{\xi M} + \left[S + \frac{M_\xi}{R_\xi} + \frac{M_\theta}{R_\theta} \right] \varepsilon_{\xi M} \right.$$
$$- \frac{1}{2E}\left[\frac{N_\xi^2}{h}\left(1 + \frac{1}{12}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi} \right)\frac{h}{R_\theta} \right) - \frac{2N_\xi M_\xi}{h^2}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi} \right) \right.$$
$$\left. + \frac{M_\xi^2}{h^3/12}\left(1 + \frac{3}{20}\left(\frac{h}{R_\theta} + \frac{h}{R_\xi} \right)\frac{h}{R_\theta} \right) \right.$$
$$+ \frac{N_\theta^2}{h}\left(1 + \frac{1}{12}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta} \right)\frac{h}{R_\xi} \right) - \frac{2N_\theta M_\theta}{h^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta} \right) \tag{14}$$
$$\left. + \frac{M_\theta^2}{h^3/12}\left(1 + \frac{3}{20}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta} \right)\frac{h}{R_\xi} \right) - 2\nu \frac{N_\xi N_\theta}{h} - 2\nu \frac{M_\xi M_\theta}{h^3/12} \right]$$
$$\left. - \frac{1}{2E_\zeta}\left[\frac{6}{5}\frac{S^2}{h} - 2\nu_\zeta \frac{S(N_\xi + N_\theta)}{h} \right] - \frac{1}{2G_\zeta}\frac{6}{5}\frac{Q^2}{h} \right\} \alpha r \, d\xi \, d\theta = 0.$$

The Euler equations of this variational equation are the following

$$\begin{aligned}(r\,N_\xi)' - r'N_\theta + (\alpha r/R_\xi)\,Q &= 0\\ (r\,Q)' - \alpha r\,[(N_\xi/R_\xi) + (N_\theta/R_\theta)] &= 0\\ (rM_\xi)' - r'M_\theta - \alpha r\,Q &= 0\\ S + (M_\xi/R_\xi) + (M_\theta/R_\theta) &= 0\end{aligned} \tag{15}$$

$$\begin{aligned}\varepsilon_{\xi M} &= \frac{1}{Eh}\left\{N_\xi\left[1 + \frac{1}{12}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right] - \nu N_\theta\right\} - \frac{M_\xi}{Eh^2}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right) - \frac{\nu_\zeta S}{E_\zeta h}\\ \varepsilon_{\theta M} &= \frac{1}{Eh}\left\{N_\theta\left[1 + \frac{1}{12}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right] - \nu N_\xi\right\} - \frac{M_\theta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right) - \frac{\nu_\zeta S}{E_\zeta h}\\ \kappa_\xi + \frac{\varepsilon_{\xi M}}{R_\xi} &= \frac{12}{Eh^3}\left\{M_\xi\left[1 + \frac{3}{20}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right] - \nu M_\theta\right\} - \frac{N_\xi}{Eh^2}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right)\\ \kappa_\theta + \frac{\varepsilon_{\xi M}}{R_\theta} &= \frac{12}{Eh^3}\left\{M_\theta\left[1 + \frac{3}{20}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right] - \nu M_\xi\right\} - \frac{N_\theta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right)\\ \varepsilon_{\zeta M} &= \frac{1}{E_\zeta h}\left[\frac{6}{5}S - \nu_\zeta\,(N_\xi + N_\theta)\right], \qquad \gamma_{\zeta M} = \frac{1}{G_\zeta h}\frac{6}{5}Q.\end{aligned} \tag{16}$$

Equations (15) include the appropriate three equilibruim equations for stress resultants and couples and a fourth equilibrium equation for the average transverse normal stress. Since this fourth equation is not a differential equation it may be used to eliminate S from the stress strain relations (16). If this is done and we also eliminate $\varepsilon_{\zeta M}$ from the third and fourth of equations (16) we are left with the following system of stress strain relations

$$\begin{aligned}\varepsilon_{\xi M} &= \frac{1}{Eh}\left\{N_\xi\left[1 + \frac{1}{12}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right] - \nu N_\theta\right\}\\ &\qquad - \frac{M_\xi}{Eh^2}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right) + \frac{\nu_\zeta}{E_\zeta h^2}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right)\\ \varepsilon_{\theta M} &= \frac{1}{Eh}\left\{N_\theta\left[1 + \frac{1}{12}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right] - \nu N_\xi\right\}\\ &\qquad - \frac{M_\theta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right) + \frac{\nu_\zeta}{E_\zeta h^2}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right)\\ \kappa_\xi &= \frac{12}{Eh^3}\left\{M_\xi\left[1 + \frac{3}{20}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right] - \nu M_\theta\right\}\\ &\quad + \frac{6}{5E_\zeta h^3}\frac{h}{R_\xi}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right) - \frac{N_\xi}{Eh^2}\left(\frac{h}{R_\theta} - \frac{h}{R_\xi}\right) + \frac{\nu_\zeta}{E_\zeta h^2}\frac{h}{R_\xi}(N_\xi + N_\theta)\\ \kappa_\theta &= \frac{12}{Eh^3}\left\{M_\theta\left[1 + \frac{3}{20}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right] - \nu M_\xi\right\}\\ &\quad + \frac{6}{5E_\zeta h^3}\frac{h}{R_\theta}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right) - \frac{N_\theta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\theta}\right) + \frac{\nu_\zeta}{E_\zeta h^2}\frac{h}{R_\theta}(N_\xi + N_\theta)\\ \gamma_{\zeta M} &= \frac{6}{5G_\zeta h}Q.\end{aligned} \tag{17}$$

The following observations may be made in regard to these stress strain relations.

(1) If we set $\nu_\zeta = 0$, $G_\zeta = \infty$ and express the N's and M's in terms of the ε's and κ's we obtain the results of Flügge [3] and Byrne [4].

(2) The terms with ν_ζ are of the same order of magnitude as some of the correction terms retained by Flügge and Byrne.

(3) Love's second approximation [2, p. 25] contains terms which are not included in the above system of relations and which are of the same order of magnitude as the present ν_ζ-terms [2, p. 28]. The reason for this is, as we shall see, that additional terms should be included in the approximations for ε_ζ and σ_ζ.

(4) Regardless of the relative magnitudes of the N's and M's we may, as long as $h/R \ll 1$, replace the terms enclosed in brackets in Equations (17) by unity.

(5) For media with pronounced transverse normal stress deformability (ν_ζ/E_ζ, $1/E_\zeta \gg 1/E$) the secondary terms involving ν_ζ and E_ζ will be more important than those involving E and may in fact become as important as the primary terms involving E.

(6) If the order of magnitude of the bending stresses is the same as that of the direct stresses, meaning that the order of magnitude of M/h is the same as that of N, and if $h/R \ll 1$ and ν_ζ/E_ζ and $1/E_\zeta$ are of the order of $1/E$, then all terms on the right of the first four equations of (17) which have h/R or $(h/R)^2$ as a factor are small compared with the terms which do not have this factor and which by themselves alone are the terms which occur in the stress strain relations of Love's first approximation. In this sense the stress strain relations of Love's first approximation, apart from the question of transverse shear stress deformability, are all that are needed for the application of the theory to specific problems.

5. Improved approximate stress strain relations. There are two reasons for wanting improvements over the stress strain relations obtained in the preceding section. (1) We found that certain terms of Love's second approximations were not contained in our results. (2) Earlier work on sandwich type shells [6] by a different method indicates that an odd contribution of the transverse normal stress might be as important as the even contribution.

In view of these facts we now wish to see to what extent replacing σ_ζ of equation (7) by the expression

$$\sigma_\zeta = \frac{\dfrac{S}{2h/3}\left[1 - \left(\dfrac{\zeta}{h/2}\right)^2\right] + \dfrac{T}{h^2}\zeta\left[1 - \left(\dfrac{\zeta}{h/2}\right)^2\right]}{(1 + \zeta/R_\xi)(1 + \zeta/R_\phi)} \tag{18}$$

has an effect on the relations between the ε's, κ's, N's and M's.

In order that the additional stress due to T can contribute to the deformation of the shell we must now also adopt an appropriately generalised expression for the displacement W. We replace the second of equations (8) by

$$W = W_M + \zeta\varepsilon_{\zeta M} + \tfrac{1}{2}\zeta^2\omega_M \tag{19}$$

where ω_M is a function of ξ only.

Introduction of (18) and (19) into the variational equation (11) results in the addition of the following integral to the variational equation (14)

$$\delta \iint \left\{ \frac{h^2}{24}\left(\frac{N_\xi}{R_\xi}+\frac{N_\theta}{R_\theta}\right)\omega_M + \frac{h}{30}T\omega_M + \frac{h^2}{40}Q\frac{1}{a}\frac{\partial \omega_M}{\partial \xi} - \frac{1}{2E_\xi}\left[\frac{2}{105}\frac{T^2}{h^2} - 2\nu_\xi\frac{2}{5}\frac{T(M_\xi+M_\theta)}{h^2}\right]\right\} \alpha r\, d\xi\, d\theta.$$

We now proceed as in going from (14) to (15) and (16). As results we have again the four equilibrium equations (15) and in addition to this a fifth equilibrium equation of the form

$$\alpha r\left[\frac{1}{30}T + \frac{h}{24}\left(\frac{N_\xi}{R_\xi}+\frac{N_\theta}{R_\theta}\right)\right] - \frac{h}{40}(rQ)' = 0. \tag{20}$$

We may use the second of equations (15) to write (20) in the alternate form

$$(T/2h) + [(N_\xi/R_\xi) + (N_\theta/R_\theta)]. \tag{20'}$$

Modifications of the stress strain relations (17) are as follows. We replace $\varepsilon_{\xi M}$ by $\varepsilon_{\xi M} + (h^2/24)(\omega_M/R_\xi)$, $\varepsilon_{\theta M}$ by $\varepsilon_{\theta M} + (h^2/24)(\omega_M/R_\theta)$ and $\gamma_{r M}$ by $\gamma_{r M} + (h^2/40)(\omega_M'/\alpha)$. We have the additional relation

$$\omega_M = \frac{1}{E_r h}\left[\frac{4}{7}\frac{T}{h} - 12\nu_r\frac{M_\xi+M_\theta}{h^2}\right] \tag{21}$$

or in view of (20')

$$\omega_M = -\frac{1}{E_r h}\left[\frac{8}{7}\left(\frac{N_\xi}{R}+\frac{N_\theta}{R_\theta}\right) + 12\nu_r\frac{M_\xi+M_\theta}{h^2}\right]. \tag{21'}$$

Finally we subtract from the expressions for κ_ξ and κ_θ the term $(2/5)(\nu_r/E_r)(T/h^2)$.

With T and ω_M of (20') and (21') the strain quantities ε and κ finally assume the following form

$$\begin{aligned}
\varepsilon_{\xi M} &= \frac{1}{Eh}\left\{N_\xi\left[1+\frac{1}{12}\left(\frac{h}{R_\theta}-\frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right] - \nu N_\theta\right\} - \frac{M_\xi}{Eh^2}\left(\frac{h}{R_\theta}-\frac{h}{R_\xi}\right) \\
&+ \frac{\nu_r}{E_r h^2}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right) + \frac{1}{E_r h}\frac{h}{R_\xi}\left[\frac{1}{21}\left(\frac{h}{R_\xi}N_\xi + \frac{h}{R_\theta}N_\theta\right) + \frac{\nu_r}{2}\frac{M_\xi+M_\theta}{h}\right] \\
\varepsilon_{\theta M} &= \frac{1}{Eh}\left\{N_\theta\left[1+\frac{1}{12}\left(\frac{h}{R_\xi}-\frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right] - \nu N_\xi\right\} - \frac{M_\theta}{Eh^2}\left(\frac{h}{R_\xi}-\frac{h}{R_\theta}\right) \\
&+ \frac{\nu_r}{E_r h^2}\left(\frac{h}{R_\xi}M_\xi + \frac{h}{R_\theta}M_\theta\right) + \frac{1}{E_r h}\frac{h}{R_\theta}\left[\frac{1}{21}\left(\frac{h}{R_\xi}N_\xi + \frac{h}{R_\theta}N_\theta\right) + \frac{\nu_r}{2}\frac{M_\xi+M_\theta}{h}\right]
\end{aligned} \tag{22}$$

$$\kappa_\xi = \frac{12}{Eh^3}\left\{M_\xi\left[1+\frac{3}{20}\left(\frac{h}{R_\theta}-\frac{h}{R_\xi}\right)\frac{h}{R_\theta}\right]-\nu M_\theta\right\}-\frac{N_\xi}{Eh^2}\left(\frac{h}{R_\theta}-\frac{h}{R_\xi}\right)$$
$$+\frac{6}{5E_\zeta h^3}\frac{h}{R_\xi}\left(\frac{h}{R_\xi}M+\frac{h}{R_\theta}M_\theta\right)+\frac{\nu_\zeta}{E_\zeta h^2}\frac{h}{R_\xi}(N_\xi+N_\theta)$$
$$+\frac{4}{5}\frac{\nu_\zeta}{E_\zeta h^2}\left(\frac{h}{R_\xi}N_\xi+\frac{h}{R_\theta}N_\theta\right)$$
$$\kappa_\theta = \frac{12}{Eh^3}\left\{M_\theta\left[1+\frac{3}{20}\left(\frac{h}{R_\xi}-\frac{h}{R_\theta}\right)\frac{h}{R_\xi}\right]-\nu M_\xi\right\}-\frac{N_\theta}{Eh^2}\left(\frac{h}{R_\xi}-\frac{h}{R_\theta}\right) \qquad (22')$$
$$+\frac{6}{5E_\zeta h^3}\frac{h}{R_\theta}\left(\frac{h}{R_\xi}M_\xi+\frac{h}{R_\theta}M_\theta\right)+\frac{\nu_\zeta}{E_\zeta h^2}\frac{h}{R_\theta}(N_\xi+N_\theta)$$
$$+\frac{4}{5}\frac{\nu_\zeta}{E_\zeta h^2}\left(\frac{h}{R_\xi}N_\xi+\frac{h}{R_\theta}N_\theta\right).$$

These four relations must be complemented by the expression for the transverse shearing strain which now reads

$$\gamma_{\xi M} = \frac{6}{5}\frac{Q}{hG_\zeta}+\frac{1}{\alpha}\left[\frac{1}{35E_\zeta}\left(\frac{h}{R_\xi}N_\xi+\frac{h}{R_\theta}N_\theta\right)+\frac{3\nu_\zeta}{10E_\zeta}\frac{M_\xi+M_\theta}{h}\right]'. \qquad (23)$$

A comparison of equations (22) and (17) shows that the choice of more flexible expressions for σ_ζ and W does in fact bring in additional terms of the same order of magnitude as those previously included.

The following further observations may be made.

(1) Equations (22) include the stress strain relations of Love's second approximation as a special case. In order to obtain these it must be assumed that $N \ll M/h$ whence

$$\varepsilon_{\xi M} = \frac{N_\xi-\nu N_\theta}{Eh}-\frac{M_\xi}{Eh^2}\left(\frac{h}{R_\theta}-\frac{h}{R_\xi}\right)$$
$$+\frac{\nu_\zeta}{E_\zeta h^2}\left[\left(\frac{h}{R_\xi}M_\xi+\frac{h}{R_\theta}M_\theta\right)+\frac{1}{2}\frac{h}{R_\xi}(M_\xi+M_\theta)\right] \qquad (24)$$

with a corresponding expression for $\varepsilon_{\theta M}$. If it is further assumed that $(h/R)^2$ can be neglected compared to unity then

$$\kappa_\xi = \frac{12}{Eh^3}(M_\xi-\nu M_\theta) \qquad (25)$$

with a corresponding expression for κ_θ.

Solution of (24) and (25) for N_ξ, N_θ, M_ξ, and M_θ makes the results appear in the same form as given in [2, p. 28]*

(2) In order to obtain stress strain relations of the same form as those for sandwich-type shells [6] where $E_\zeta \ll E$ it is necessary to omit in (22) all terms involving factors h/R except those involving E_ζ and to set $\nu_\zeta = 0$.

* In Equation (54) of [2] read $\frac{1}{2}(\kappa_1+\kappa_2)$ instead of $(\kappa_1+\kappa_2)$.

(3) In order to judge the differences between the expression for $\gamma_{\xi M}$ according to (17) and (23) we may write (23) with the help of the second of equations (15) in the alternate form

$$\gamma_{\xi M} = \frac{6}{5}\frac{Q}{hG_\xi} + \frac{h}{35E_\xi}\frac{1}{\alpha}\left[\frac{(rQ)'}{\alpha r}\right]' + \frac{3}{10}\frac{\nu_\xi}{E_\xi h}\frac{(M_\xi + M_\vartheta)'}{\alpha}. \tag{23'}$$

As long as significant changes of Q occur over distances only which are large compared with the shell thickness h we may omit the second term in (23'). The same is however not necessarily true for the third term in (23'). Consequently we shall take as transverse shear stress strain relation

$$\gamma_{\xi M} = \frac{6}{5}\frac{Q}{hG_\xi} + \frac{3}{10}\frac{\nu_\xi}{E_\xi h}\frac{(M_\xi + M_\vartheta)'}{\alpha} \tag{23''}$$

(4) If in (22) we neglect second order corrections in h/R in comparison with first order corrections and assume that E_ξ and E are of the same order of magnitude then we are left with folloing system of stress strain relations

$$\begin{aligned}
\varepsilon_{\xi M} &= \frac{N_\xi - \nu N_\vartheta}{Eh} - \frac{M_\xi}{Eh^2}\left(\frac{h}{R_\vartheta} - \frac{h}{R_\xi}\right) + \frac{\nu_\xi}{E_\xi h^2}\left[\frac{3}{2}\frac{h}{R_\xi}M_\xi + \left(\frac{h}{R_\vartheta} + \frac{1}{2}\frac{h}{R_\xi}\right)M_\vartheta\right] \\
\kappa_\xi &= \frac{12(M_\xi - \nu M_\vartheta)}{Eh^3} - \frac{N_\xi}{Eh^2}\left(\frac{h}{R_\vartheta} - \frac{h}{R_\xi}\right) + \frac{\nu_\xi}{E_\xi h^2}\left[\frac{9}{5}\frac{h}{R_\xi}N_\xi + \left(\frac{h}{R_\xi} + \frac{4}{5}\frac{h}{R_\vartheta}\right)N_\vartheta\right] \\
\kappa_\vartheta &= \frac{12(M_\vartheta - \nu M_\xi)}{Eh^3} - \frac{N_\vartheta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\vartheta}\right) + \frac{\nu_\xi}{E_\xi h^2}\left[\frac{9}{5}\frac{h}{R_\vartheta}N_\vartheta + \left(\frac{h}{R_\vartheta} + \frac{4}{5}\frac{h}{R_\xi}\right)N_\xi\right] \\
\varepsilon_{\vartheta M} &= \frac{N_\vartheta - \nu N_\xi}{Eh} - \frac{M_\vartheta}{Eh^2}\left(\frac{h}{R_\xi} - \frac{h}{R_\vartheta}\right) + \frac{\nu_\xi}{E_\xi h^2}\left[\frac{3}{2}\frac{h}{R_\vartheta}M_\vartheta + \left(\frac{h}{R_\xi} + \frac{1}{2}\frac{h}{R_\vartheta}\right)M_\xi\right].
\end{aligned} \tag{26}$$

6. Concluding remarks. We have not succeeded in the foregoing in arriving at a system of stress strain relations which we know for sure contains all correction terms of order h/R to the stress-strain relations of Love's first approximation. In view of the assumptions (7), (8), and (18) for stresses and displacements it is however possible to think that all *significant* correction terms have been obtained.

There are difficulties in going further, since it appears that when more flexible approximations for stresses and displacements are chosen then the final stress strain relations (26) are always replaced by relations which contain besides resultants and couples also *derivatives* of such quantities and higher moments of the stresses [2]. These higher order moments mean that we are then outside the realm of what we generally understand as shell theory.

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THE BROWNIAN MOTION OF COUPLED SYSTEMS[1,2]

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1. Introduction. We consider a mechanical *instrument*[3] which is surrounded by a fluid (gas or liquid), and is in a state of macroscopic equilibrium. Due to the impact of the molecules of the surrounding fluid, the instrument will have a certain random motion which, in the case of small particles, is commonly known as Brownian motion. In the present paper we are concerned with obtaining a measure of this random motion.

We assume that the instrument and a finite number of fluid molecules are enclosed in a motionless container, and that the *system* consisting of the instrument and the molecules is in thermodynamic equilibrium, i.e., there is no flow of heat through the surrounding container. We define an *ensemble* as a large number of such systems, each of which have the same *macroscopic* properties as the given system, but which assume all possible internal configurations. We shall sidestep the delicate questions of ergodic theory (see, for instance [4] or [5]), and merely state that by definition the systems in the ensemble are distributed in such a way that the proportion of systems which experience a certain event in a time interval Δt is identical with the probability that the given system will experience this event in the interval Δt. We then define the average value of a parameter of the system as the average of the parameter over all systems in the ensemble.

Let the generalized coordinates of the instrument be denoted by $x_1, \cdots, x_n$. We shall, in general, refer to the array of these coordinates by the single letter x, and to a particular coordinate by x_i. The velocities will be similarly denoted by $\dot{x}$ or $\dot{x}_i$, and the conjugate momenta by p or p_i. The positions, velocities, and conjugate momenta of the fluid molecules will be denoted by x', $\dot{x}'$, and p', respectively.

If the system is in thermodynamic equilibrium, its coordinates will satisfy the Maxwell-Boltzmann distribution law [4, 5, 6]. Accordingly, the average value of a function of the coordinates and conjugate momenta will be given by

$$\overline{W(x, x', p, p')} = \frac{\iiiint W(x, x', p, p') e^{-H^*/kT}\, dx\, dx'\, dp\, dp'}{\iiiint e^{-H^*/kT}\, dx\, dx'\, dp\, dp'}. \tag{1.1}$$

Here H^* is the Hamiltonian of the system expressed as a function of the coordinates and momenta, k is the Boltzmann gas constant, and T is the absolute

[1] The results presented in this paper were obtained in the course of research conducted for North American Aviation, Inc.

[2] Numbers in square brackets refer to the bibliography at the end of the paper.

[3] We use the term "instrument" to refer to any mechanical system, reserving the term "system" to refer to the instrument together with the surrounding fluid, in accord with the usual usage of statistical mechanics.

temperature. The integration is to be carried out over all admissible values of the variables. If we assume that the energy of interaction of the instrument and the molecules is negligible compared to the individual energies of either, we may write $H^* = H + H' + H'' \cong H + H'$, where H is the Hamiltonian of the instrument, H' that of the molecules, and H'' that of the interaction. It follows that for functions of the instrument variables only, the average value is given by

$$\overline{W(x, p)} = \frac{\iint W(x, p)e^{-H(x,p)/kT}\, dx\, dp}{\iint e^{-H(x,p)/kT}\, dx\, dp}. \tag{1.2}$$

Our primary problem is to determine a measure of the displacement of the instrument from its initial equilibrium position, due to the random impacts of the fluid molecules. In the case where the instrument is in stable equilibrium, such a measure may be calculated directly from Eq. 1.2. The average $\bar{x}$ will simply give the equilibrium position of the instrument, and can always be taken to be zero. However, the average $\overline{x^2}$ will of necessity be positive, and its magnitude will provide a meaningful measure. In Sec. 2 we shall carry out a series of transformations on the expression for the energy of the instrument, which will enable us to reduce the calculation of any average of a function of position only to the evaluation of definite integrals. In particular, we shall obtain an expression for the mean-square displacement.

For the case of neutral equilibrium, the expressions $\overline{x^2}$ would all be infinite, since it is intuitively obvious that an instrument in neutral equilibrium could undergo an arbitrarily large displacement, provided only that the elapsed time is sufficiently great. In this case, any meaningful measure of the displacement will, of course, be time dependent.

In Sec. 3 we shall define such a measure, and state the conditions under which the present method will lead to a solution.

In Sec. 4, we shall compute the average values of certain functions of velocity only, and in Sec. 5 we shall use these results to find the desired measure of displacement. Finally, in Sec. 6, we shall consider some limitations of our results.

2. Stable equilibrium. The potential energy of the instrument is a function of position only, while the kinetic energy can always be written in the form

$$K(x, \dot{x}) = \sum_{i,j} a_{ij}(x)\dot{x}_i\dot{x}_j + \sum_i b_i(x)\dot{x}_i + c(x). \tag{2.1}$$

Here the indices to be summed are indicated by the subscripts, and the summations are all from one to n.

We assume that for any real mechanical instrument the quadratic part of K will be positive definite. Equation 2.1 can then be reduced to a sum of squares by the successive transformations

$$\dot{x}_i = \zeta_i + h_i \tag{2.2}$$

and

$$\zeta_i = \sum_r l_{ir}\eta_r , \tag{2.3}$$

where the h_i are defined by

$$2\sum_k a_{ik}h_k + b_k = 0, \tag{2.4}$$

and the l_{ir} by

$$\sum_j l_{ij}l_{rj} = \sum_j l_{ji}l_{jr} = \delta_{ir} = \begin{cases} 0 & \text{if } i \neq r, \\ 1 & \text{if } i = r, \end{cases} \tag{2.5}$$

$$\sum_j a_{ij}l_{im}l_{jn} = 0 \qquad \text{if } m \neq n.$$

The kinetic energy 2.1 may be then written

$$K(x, \dot{x}) = K^*(x, \eta) = \sum_m e_m\eta_m^2 + d, \tag{2.6}$$

where

$$d = d(x) = K[x, h(x)] = \sum_{ij}a_{ij}h_ih_j + \sum_i b_ih_i + c, \tag{2.7}$$

and

$$e_m = e_m(x) = \sum_{ij}a_{ij}l_{im}l_{jm}. \tag{2.8}$$

The conjugate momenta p may be expressed in terms of the coordinates η by means of

$$p_k = \frac{\partial K}{\partial \dot{x}_k} = \sum_{ij}\frac{\partial K^*}{\partial \eta_i}\frac{\partial \eta_i}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \dot{x}_k} = 2\sum_i e_i l_{ki}\eta_i . \tag{2.9}$$

If the linear equations 2.9 are solved for the η_i, and the results substituted into 2.6, we obtain

$$K^{**}(x, p) = K^*[x, \eta(x, p)]. \tag{2.10}$$

The potential energy is at most a function of position, so that the Hamiltonian occurring in Eq. 1.2 is given by

$$H(x, p) = K^{**}(x, p) + V(x). \tag{2.11}$$

However, we are interested in functions of the velocities, rather than of the momenta. Therefore, let us transform the integrations of Eqs. 1.2 into integrations over the η's by means of Eqs. 2.9. Since the Jacobian of this transformation is the determinant

$$J(\rho/\eta) = |\,\partial\rho_i/\partial\eta_j\,| = 2\,|\,e_j l_{ij}\,|, \tag{2.12}$$

the resulting formula is

$$\overline{W(x, \eta)} = \frac{\iint W(x, \eta)2\,|\,e_j l_{ij}\,| \exp\{-[\sum_j e_j\eta_j^2 + d + V]/kT\}\,dx\,d\eta}{\iint 2\,|\,e_j l_{ij}\,| \exp\{-[\sum_j e_j\eta_j^2 + d + V]/kT\}\,dx\,d\eta}. \tag{2.13}$$

Finally, we make the transformation

$$\xi_i^2 = e_i\eta_i^2/kT \tag{2.14}$$

whose Jacobian is

$$J(\eta/\xi) = |\,\partial\eta_i/\partial\xi_j\,| = (kT)^{n/2}/e_1e_2\cdots e_n\,. \tag{2.15}$$

Defining

$$f(x) = |\,e_j\delta_{ij}\,|\cdot(kT)^{n/2}/e_1e_2\cdots e_n\,, \tag{2.16}$$

we may now write Eq. 2.13 in the form

$$\overline{W(x,\xi)} = \frac{\iint W(x,\xi)f(x)\exp\{-\sum_i\xi_i^2 - (d+V)/kT\}\,dx\,d\xi}{\iint f(x)\exp\{-\sum_i\xi_i^2 - (d+V)/kT\}\,dx\,d\xi}\,. \tag{2.17}$$

In Eq. 2.17, the displacement variables x and the "velocity" variables ξ, may now be separated, provided that W is separable. In particular, if W depends on only displacement or velocity, we have

$$\overline{W(x)} = \frac{\int W(x)f(x)e^{-(d+V)/kT}\,dx}{\int f(x)e^{-(d+V)/kT}\,dx}\,, \tag{2.18}$$

and

$$\overline{W(\xi)} = \frac{\int W(\xi)e^{-(\xi_1^2+\ldots+\xi_n^2)}\,d\xi}{\int e^{-(\xi_1^2+\ldots+\xi_n^2)}\,d\xi}\,. \tag{2.19}$$

In particular, the mean-square displacement of the ith coordinate is given by

$$\overline{x_i^2} = \frac{\int x_i^2 f(x)e^{-(d+V)/kT}\,dx}{\int f(x)e^{-(d+V)/kT}\,dx}\,. \tag{2.20}$$

3. Neutral equilibrium. At time $t = 0$, let the displacement of the ith coordinate of a particular instrument be x_i^1, and of some later time t let it be $x_i^1 + \Delta x_i$. We shall define our desired measure of displacement to be $\overline{\Delta x_i^2}$. As previously noted, this displacement will be a function of t.

We shall be forced to draw a rather fine line as to the length of the time interval t which we are allowed to consider. In the derivation of our results, we shall consider only a "mathematically small" time interval in that we shall reserve the right to neglect high powers of t. The maximum time interval for which this procedure will be valid must be investigated in relation to any particular prob-

lem. On the other hand, as we shall show in Sec. 6, any physical interpretation of our results will be valid only for a sufficiently large time interval. There does not seem to be any general method of showing that these two requirements are not mutually exclusive. The author can only state that in the particular application which led to the present investigation there was a useful time interval which satisfied both requirements.

Several restrictions must also be placed on the expression 2.1 for the kinetic energy. We observe first that the functions $a_{ij}(x)$ and $b_i(x)$ may be regarded as functions of time for any particular motion. We require then, that at some fixed time $t = t_0$ the kinetic energy be orthogonal and symmetric with respect to the coordinates x, i.e., that

$$b_i = 0, \qquad a_{ij} = 0 \qquad \text{if } i \neq j. \tag{3.1}$$

Thus at time $t = t_0$ the kinetic energy may be written in the form of Eq. 2.6 in the original velocity variables $\dot{x}_i$. Further, we demand that at time $t = t_0$ the a_{ij} satisfy the equations

$$\frac{\partial}{\partial x_m}\left(\frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{ik}}{\partial x_j} - \frac{\partial a_{jk}}{\partial x_i}\right) = 0. \tag{3.2}$$

A sufficient condition for this is that the a_{ij} are linear in x. Finally, the variation of a_{ij} and b_i with the actual motion must be such that for the time interval $(0, t)$ under consideration they may be written as linear functions of x.

Under these restrictions, we shall show that the mean-square displacement is a linear function of t, and shall describe a procedure for evaluating the coefficients. For this purpose, we shall need the averages of all linear, quadratic, and cubic terms in the velocity variables. The following section is devoted to that computation.

4. Averages of functions of velocity only. We shall first compute the corresponding averages of the functions ξ_i by means of Eq. 2.19. To this end, we must define our limits of integration. The actual possible values of the velocities will in general be limited by physical considerations (for example, a linear velocity must be less than the speed of light). However, these limits will be so large compared to velocities introduced by the impact of the molecules, that we will be justified in taking them as infinite. It follows, then, that the limits of integration on ξ will also be plus and minus infinity. For example,

$$\bar{\xi}_1 = \frac{\int_{-\infty}^{\infty} \xi_1 e^{-\xi_1^2} \, d\xi_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(\xi_2^2 + \ldots + \xi_n^2)} \, d\xi_2 \cdots d\xi_n}{\int_{-\infty}^{\infty} e^{-\xi_1^2} \, d\xi_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(\xi_2^2 + \ldots + \xi_n^2)} \, d\xi_2 \cdots d\xi_n}.$$

Since the numerator integral of the above expression is odd in ξ_1, while none of the denominator integrals vanish, we obtain $\bar{\xi}_1 = 0$. Similarly, it is easily shown that

$$\bar{\xi}_i = 0, \qquad \overline{\xi_i \xi_j} = 0 \qquad \text{if } i \neq j, \qquad \text{and } \overline{\xi_i \xi_j \xi_k} = 0. \tag{4.1a}$$

On the other hand, if $i = j$, we have

$$\overline{\xi_i^2} = \int_{-\infty}^{\infty} \xi_i^2 e^{-\xi_i^2}\, d\xi_i \Big/ \int_{-\infty}^{\infty} e^{-\xi_i^2}\, d\xi_i = \frac{1}{2}. \tag{4.1b}$$

We will now express the desired velocity averages in terms of the averages computed in Eqs. 4.1. In view of Eqs. 2.2, 2.3, and 2.14, we may write

$$\dot{x}_i = \sum_r l_{ir}(kT/e_r)^{\frac{1}{2}}\xi_r + h_i\,. \tag{4.2}$$

Since l_{ir}, e_r, and h_i are all functions of x only, while ξ_r is independent of x, the average of Eqs. 4.2 over the ensemble yields

$$\bar{\dot{x}}_i = \sum_r \overline{l_{ir}(kT/e_r)^{\frac{1}{2}}}\,\bar{\xi}_r + \bar{h}_i = \bar{h}_i, \tag{4.3a}$$

where the last step follows from the first Eq. 4.1. A similar analysis applied to the quadratic and cubic velocity terms yields

$$\overline{\dot{x}_i\dot{x}_j} = (kT/2)\sum_r \overline{(l_{ir}l_{ir}/e_r)} + \overline{h_i h_j}, \tag{4.3b}$$

and

$$\overline{\dot{x}_i\dot{x}_j\dot{x}_k} = (kT/2)\sum_r \overline{(l_{ir}l_{jr}h_k + l_{jr}l_{kr}h_i + l_{kr}l_{ir}h_j)/e_r} + \overline{h_i h_j h_k}. \tag{4.3c}$$

In Eqs. 4.3, the functions h_i, l_{ij}, and e_j, are defined by Eqs. 2.4, 2.5, 2.8, respectively, and the averages are to be completed by Eq. 2.18. To this end, we observe that to first order terms in x_i, any function $\phi(x)$ may be written

$$\phi(x_1, \cdots, x_n) = \phi(0, \cdots, 0) + \sum_m \frac{\partial\phi}{\partial x_m}(0, \cdots, 0)x_m. \tag{4.5}$$

We shall use expressions of the form occurring in Eq. 4.5 so frequently that it seems desirable to introduce a simplified notation. Let

$$\phi^0 = \phi(0, \cdots, 0), \qquad \phi_{,\,m} = \partial\phi/\partial x_m\,. \tag{4.6}$$

Then Eq. 4.5 may be written

$$\phi(x) = \phi^0 + \sum_m \phi^0_{,\,m}\, x_m\,. \tag{4.7}$$

In view of our assumptions (3.1) of initial orthogonality and symmetry, the functions a_{ij} and b_i may now be written

$$\begin{aligned} a_{ii} &= a^0_{ii} + \textstyle\sum_k a^0_{ii,\,k}\, x_k\,, \\ a_{ij} &= \textstyle\sum_k a^0_{ij,\,k}\, x_k \qquad\qquad \text{for } i \neq j, \\ b_i &= \textstyle\sum_k b^0_{i,\,k}\, x_k\,. \end{aligned} \tag{4.8a}$$

Further, the zero order terms of h_k and l_{ij} must be such as to preserve the symmetry and orthogonality, so that

$$\begin{aligned} h_k &= \textstyle\sum_m h^0_{k,m}\, x_m\,, \\ l_{ij} &= \delta_{ij} + \textstyle\sum_m l^0_{ij,m}\, x_m\,. \end{aligned} \tag{4.8b}$$

Substituting Eqs. 4.8 into Eqs. 2.7, 2.8, and 2.16, we obtain

$$d = c, \qquad e_k = a^0_{kk} + \sum_m e^0_{k,m}\, x_m\,, \qquad f = 2(kT)^{n/2} + \sum_m f^0_{,m}\, x_m\,, \tag{4.9}$$

to first order terms.

Therefore, Eq. 2.18 for computing averages of functions of x becomes

$$\overline{W(x)} = \frac{\int W(x)[2(kT)^{n/2} + \sum_r f^0_{,r}\, x_r]e^{-\epsilon/kT}\, dx}{\int [2(kT)^{n/2} + \sum_r f^0_{,r}\, x_r]e^{-\epsilon/kT}\, dx}\,. \tag{4.10}$$

Now, in view of Eq. 4.7, we are interested only in linear functions of x. Neglecting powers of x higher than the first, we have

$$\begin{aligned}\overline{W(x)} &= \overline{W^0} + \sum_r \overline{W^0_{,r}\, x_r} \\ &= \frac{\int [W^0 2(kT)^{n/2} + W^0 \sum_r f^0_{,r} x_r + 2(kT)^{n/2} \sum_r W^0_{,r} x_r]e^{-\epsilon/kT}\, dx}{\int [2(kT)^{n/2} + \sum_r f^0_{,r} x_r]e^{-\epsilon/kT}\, dx} \\ &= W^0,\end{aligned} \tag{4.11}$$

where the last step follows from the assumed symmetry of x. In other words, the average value of a function of x will differ from its value at the origin by terms of second or higher order. In particular, then, we can evaluate the averages occurring in Eqs. 4.3, and so evaluate the desired velocity averages. The results are easily seen to be

$$\overline{x_i} = 0, \qquad \overline{x_i x_j} = (kT/2a^0_{ii})\delta_{ij}\,, \qquad \overline{x_i x_j x_k} = 0. \tag{4.12}$$

5. Mean square displacements for neutral equilibrium. We are now ready to find the mean-square displacements for neutral equilibrium. Since the potential energy is constant, the Lagrangian equations of motion may be written

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_k}\right) - \frac{\partial K}{\partial x_k} = F_k \tag{5.1}$$

Here the F_k represent the forces of interaction between the instrument and the fluid molecules. Although these forces could be neglected in the computation of velocity averages, they must, of course, be considered in Eqs. 5.1 since otherwise we could predict no motion at all. Experimental evidence indicates that the viscous force of a fluid is proportional to the velocity[4], at least for low speeds. Therefore, we shall write F_k in the form

$$F_k = -\sum_j r_{kj}\, \dot{x}_j + g_k\,(x). \tag{5.2}$$

[4] This assumes that the time interval under discussion is sufficiently *large* so that many molecules will have collided with the instrument. Such a time will still be *small* in the sense we have used it. This subject will be discussed further in Sec. 6.

For a homogeneous isotropic fluid, the "coefficients of viscous damping" r_{kj} are constant, and may be computed from the viscosity and the shape of the instrument. The functions g_i may depend upon the coordinates x, and are not known.

Substituting Eqs. 2.1 and 5.2 into the equations of motion 5.1, we obtain

$$2\sum_m a_{km}\ddot{x}_m + \sum_{ij}\left[\frac{\partial a_{ik}}{\partial x_j} + \frac{\partial a_{jk}}{\partial x_i} - \frac{\partial a_{ij}}{\partial x_k}\right]\dot{x}_i\dot{x}_j + \sum_m\left[\frac{\partial b_k}{\partial x_m} - \frac{\partial b_m}{\partial x_k}\right]\dot{x}_k - \frac{\partial c}{\partial x_k} = \sum_m r_{km}\dot{x}_m + g_k. \tag{5.3}$$

As previously mentioned, we are considering a mathematically small time interval t. Since Δx_i tends to zero with t, it may also be considered as a small quantity. Therefore, if x_i^1 are the coordinates of an instrument at time $t = 0$, any function $\psi(x)$ of the coordinates of the instrument may be written

$$\psi(x^1 + \Delta x) = \psi^1 + \sum_r \psi^1_{,r}\,\Delta x_r, \tag{5.4}$$

where, in analogy to Eq. 4.6, we have defined

$$\psi^1 = \psi(x_1^1, x_2^1, \cdots, x_n^1). \tag{5.5}$$

Expanding the functions of x in Eq. 5.3 in this manner, we obtain the equations of motion at an arbitrary time t for the particular instrument under discussion:

$$\begin{aligned} &2\sum_m(a^1_{km} + \sum_r a^1_{km,r}\,\Delta x_r)\ddot{x}_m \\ &\quad + \sum_{ij}\left[(a^1_{ik,j} + a^1_{jk,i} - a^1_{ij,k}) + \sum_r(a^1_{ik,jr} + a^1_{jk,ir} - a^1_{ij,kr})\Delta x_r\right]\dot{x}_i\dot{x}_j \\ &\quad + \sum_m\left[(b^1_{k,m} - b^1_{m,k} + r^1_{km}) + \sum_r(b^1_{k,mr} - b^1_{m,kr} + r^1_{km,r})\Delta x_r\right]\dot{x}_m \\ &\quad - \left[c^1_{,k} + g^1_k + \sum_r(c^1_{,kr} + g^1_{k,r})\Delta x_r\right] = 0. \end{aligned} \tag{5.6}$$

Equations 5.6 are valid for each instrument in the ensemble. Therefore, we may multiply each equation by any function of the coordinates of that system and average over the ensemble. In particular, we shall multiply in turn by the quantities $\dot{x}_p$ and Δx_p. Since these are each independent of the initial coordinates x_i^1 of the instrument in question, all averages of products may be separated into products of averages as indicated at the end of Sec. 2: one factor being a function of x_i^1 only, and the other factor depending upon $\dot{x}_i$ and Δx_i. This second factor cannot be further separated, since $\dot{x}_i$ and Δx_i are not independent, but are in fact, related by

$$(d/dt)\,(\Delta x_i) = \dot{x}_i, \tag{5.7}$$

as is easily seen from our basic definitions.

Equation 5.7 is a linear relation. Therefore, any term which is odd in $\dot{x}_i$ and Δx_i combined will have an average value zero, due to the assumed symmetry. It follows that the second, third, sixth, and seventh group of terms in Eq. 5.6 will contribute nothing to the averages we are going to take, and hence may be neglected. Thus, for our purposes, we may replace Eq. 5.6 by

$$2\sum_m a^1_{km}\ddot{x}_m + \sum_m \alpha_{km}\dot{x}_m + \sum_m \beta_{km}\Delta x_m = 0, \tag{5.8}$$

where α_{km} and β_{km} are defined by

$$\begin{aligned} \alpha_{km} &= b^1_{k,m} - b^1_{m,k} + r^1_{km}, \\ \beta_{km} &= -(c^1_{,km} + g^1_{k,m}). \end{aligned} \tag{5.9}$$

Next, it follows from the first Eq. 4.8 that when we average over the ensemble the only term in the first summation of Eq. 5.8 which does not vanish is the term where $m = k$. Also, in view of Eq. 4.11, the average values of the functions of x_i^1 may be taken to be their values at the position $x_i = 0$. Thus, it follows from Eqs. 5.9 that

$$\begin{aligned} \overline{\alpha_{km}} &= b^0_{k,m} - b^0_{m,k} + r^0_{km}, \\ \overline{\beta_{km}} &= -(c^0_{,km} + g^0_{k,m}). \end{aligned} \tag{5.10}$$

We are now ready to multiply Eq. 5.6 (or equivalently, Eq. 5.8) by $\dot{x}_i$ and Δx_i in turn, the average over the ensemble. The resulting equations are

$$2a^0_{kk}\,\overline{\dot{x}_k \dot{x}_i} + \sum_m \overline{\alpha_{km}}\,\overline{\dot{x}_m \dot{x}_i} + \sum_m \overline{\beta_{km}}\,\overline{\dot{x}_i\,\Delta x_m} = 0, \tag{5.11}$$

$$2a^0_{kk}\,\overline{\dot{x}_k \Delta x_i} + \sum_m \overline{\alpha_{km}}\,\overline{\dot{x}_m \Delta x_i} + \sum_m \overline{\beta_{km}}\,\overline{\Delta x_i\,\Delta x_m} = 0. \tag{5.12}$$

The first two terms of Eq. 5.11 are independent of the displacement, and hence, in the average, will be independent of time. It follows that the solutions of Eqs. 5.11 can be written in the form

$$\overline{\dot{x}_i\,\Delta x_m} = \tfrac{1}{2}\, d_{mi} = \text{const.} \tag{5.13}$$

Infeld [3] has shown that the orders of differentiation and averaging may be interchanged. Setting $i = m$ in Eq. 5.13 and integrating, we obtain

$$\overline{\Delta x_i^2} = d_{ii} t. \tag{5.14}$$

Thus we have shown that the mean-square displacement is a linear function of time. It remains to only compute the coefficients d_{ii}.

We observe first, that in view of Eqs. 5.13, 5.7, and 4.12,

$$\overline{\Delta x_i\,\dot{x}_k} = \overline{\frac{d}{dt}(\Delta x_i\,\dot{x}_k)} - \overline{\dot{x}_k \frac{d}{dt}(\Delta x_i)} = -(kT/2a^0_{kk})\delta_{ik}. \tag{5.15}$$

Similarly,

$$\overline{\Delta x_i \Delta x_m} = \overline{\dot{x}_i \Delta x_m} + \overline{\dot{x}_m \Delta x_i} = \tfrac{1}{2}(d_{im} + d_{mi}),$$

so that

$$\overline{\Delta x_i \Delta x_m} = \tfrac{1}{2}(d_{im} + d_{mi})t. \tag{5.16}$$

Equation 5.16 shows that for arbitrarily small values of t, the last term in Eq. 5.12 may be neglected. Therefore, in view of 5.13 and 5.15, Eq. 5.12 may be written

$$\sum_m \tfrac{1}{2}\,\overline{\alpha_{km}}\, d_{im} = kT\delta_{ik} \tag{5.17}$$

The terms $\overline{\alpha_{km}}$ in Eq. 5.16 represent known functions, being defined in terms of given quantities by the first Eq. 5.9. Therefore, if the determinant $|\overline{\alpha_{km}}|$ is different from zero, the n^2 linear equations 5.17 may be solved for the n^2 unknowns d_{im}, and the results substituted into Eqs. 5.14 to give the mean-square displacements.

6. Limitations. As Einstein [7] has pointed out, the above results are not valid for very small time intervals. To see this, let us take the time t so small that there are a total of less than N collisions between a molecule and an instrument in the ensemble of N instruments. If the instrument is large compared to a single molecule, the effect of these collisions can be ignored in discussing the motion of each instrument. Therefore, if $\dot{x}_i$ is the velocity of a particular instrument at time $t = 0$, it follows from Newton's second law of motion that the displacement at the end of time t will be

$$\Delta x_i = \dot{x}_i^0 t. \tag{6.1}$$

Equation 6.1 is valid for each instrument in the ensemble. Squaring and averaging, we obtain

$$\overline{\Delta x_i^2} = \overline{(\dot{x}_i^0)^2} t^2 = (kT/2a_{ii}^0)t^2 \tag{6.2}$$

where the last step follows from Eq. 4.12. This is in obvious disagreement with Eq. 5.4, so that our previous results are not valid in this case.

As we consider increasingly long intervals t, the statistical approach to molecule impacts contained in the concept of viscosity (Eq. 5.2) becomes increasingly valid. As a measure of the minimum time for which these results are valid, let us compute the time interval for which the mean-square displacements predicated by 5.14 and 6.2 are equal. Denoting this time by t_1, we have

$$\begin{aligned} d_{ii}t_1 &= kTt_1^2/2a_{ii}^0, \\ t_1 &= 2a_{ii}^0\, d_{ii}/kT. \end{aligned} \tag{6.3}$$

Therefore, for $t > t_1$, the statistical effect of molecule bombardment will predominate over the inertia of the instrument. It is worth noting that for time intervals smaller than t_1, the mean-square displacement computed according to Sec. 5 will be greater than the actual mean-square displacement. Therefore, if, as is often the case, we are interested only in obtaining a reasonable upper bound on the mean-square displacement, we may use the results of Sec. 5 for all values of the time for which the predicted displacements are sufficiently small.

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AN ASSOCIATED BESSEL FUNCTION*

By Yudell L. Luke

1. Introduction. The purpose of this paper is to investigate the properties of an associated Bessel function. The technique follows the pattern for the development of the well-known Lommel functions[1]. The result is applied to explicitly evaluate some indefinite integrals involving Bessel functions of the type previously considered by the author[2]. These integrals are of considerable importance in applied problems and a partial list of applications is given in this reference. The present analysis is extended to the evaluation of some infinite integrals which are also of interest. Other applications are also given.

2. Solution of a differential equation. Let

$$\Delta_\nu = z^2(d^2/dz^2) + z(d/dz) - (\nu^2 + z^2) \tag{1}$$

and consider the differential equation

$$\Delta_\nu y = e^{-z} z^{\mu+1} \tag{2}$$

The particular solution of (2) in ascending powers of z is

$$y = e^{-z} z^{\mu+1} \frac{\Gamma(\mu - \nu + 1)\,\Gamma(\mu + \nu + 1}{\Gamma(\mu + \tfrac{3}{2})} \cdot \sum_{m=0}^{\infty} \frac{(2z)^m \Gamma(\mu + m + \tfrac{3}{2})}{\Gamma(\mu - \nu + m + 2)\,\Gamma(\mu + \nu + m + 2)} \tag{3}$$

Note that (3) is not defined if $\mu \pm \nu$ is a negative integer. It is finite if μ is an odd multiple of $-\frac{1}{2}$ (except $-\frac{1}{2}$) provided that both the numbers $\mu \pm \nu$ are not negative integers. For brevity, we denote the series (3) when it is defined by $h_{\mu,\nu}(z)$. It can be shown that $h_{\mu,\nu}(z)$ satisfies the following recurrence relations:

$$(2\mu + 3)h_{\mu+1,\nu}(z) + [\nu^2 - (\mu + 1)^2]h_{\mu,\nu}(z) + e^{-z} z^{\mu+1} = 0 \tag{4}$$

$$h'_{\mu,\nu}(z) + \frac{\nu}{z} h_{\mu,\nu}(z) = \frac{(\nu + \mu)(\nu + \mu - 1)}{2\mu + 1} h_{\mu-1,\nu-1}(z) + \frac{e^{-z} z^{\mu}}{2\mu + 1} \tag{5}$$

$$h'_{\mu,\nu}(z) - \frac{\nu}{z} h_{\mu,\nu}(z) = \frac{(\nu - \mu)(\nu - \mu + 1)}{2\mu + 1} h_{\mu-1,\nu+1}(z) + \frac{e^{-z} z^{\mu}}{2\mu + 1} \tag{6}$$

The particular solution of (2) in descending powers of z is given by

$$y = -e^{-z} z^{\mu+1} \frac{\Gamma(\mu - \nu + 1)\Gamma(\mu + \nu + 1)}{\Gamma(\mu + \tfrac{3}{2})} \cdot \sum_{m=0}^{\infty} \frac{\Gamma(\mu - m + \tfrac{1}{2})}{(2z)^{m+1}\Gamma(\mu - \nu - m + 1)\Gamma(\mu + \nu - m + 1)} \tag{7}$$

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The series given by (7), is not convergent unless it terminates, and the series is composed of a finite number of terms if $\mu \pm \nu$ is a positive integer or zero. In this instance, the finite series is a solution of (2) and we designate it by $H_{\mu,\nu}(z)$. Evidently the expansion (7) is not defined if $\mu = -\frac{1}{2}$. The same is true if μ is an odd multiple of $\frac{1}{2}$ unless both the numbers $\mu \pm \nu$ are positive integers or zero. As in the case for Lommel functions, the series (7) is of significance when it is defined and consists of an infinite number of terms, for then it yields an asymptotic expansion of $H_{\mu,\nu}(z)$ which is valid if z is sufficiently large. It is easy to verify that $H_{\mu,\nu}(z)$ can replace $h_{\mu,\nu}(z)$ in the recurrence relations (4)–(6).

We interrupt further discussion of the function $H_{\mu,\nu}(z)$ to record some expressions which prove useful for our later work. Application of the duplication formula for Gamma functions[3] to a well-known result[4] yields

$$I_\nu(z) = \frac{e^{-z}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(2z)^{\nu+m}\Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(2\nu + m + 1)}. \tag{8}$$

Employing (8) and the definition of the modified Bessel function of the second kind, one can derive that

$$\begin{aligned} K_n(z) = {} & \frac{e^{-z}}{\sqrt{\pi}} \sum_{m=0}^{2n-1} (-2z)^{m-n}(2n - m - 1)!\Gamma(-n + m + \tfrac{1}{2}) \\ & + (-)^{n+1} I_n(z) \log 2z + (-)^n \frac{e^{-z}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(2z)^{m+n}\Gamma(m + n + \frac{1}{2})}{m!(m + 2n)!} \\ & \cdot \{\psi(m + 1) - \psi(m + n + \tfrac{1}{2}) + \psi(2n + m + 1)\} \end{aligned} \tag{9}$$

Here $\psi(z)$ is the logarithmic derivative of the Gamma function.

3. The general definition of $H_{\mu,\nu}(z)$. To return to the function $H_{\mu,\nu}(z)$, suppose that $(\mu - \nu)$ is a positive integer or zero. Employing (3), (7) and (8), a straightforward reduction shows that

$$H_{\mu,\nu}(z) = h_{\mu,\nu}(z) - \sqrt{\pi}\, \frac{\Gamma(\mu - \nu + 1)\Gamma(\mu + \nu + 1)}{2^{\mu+1}\Gamma(\mu + \frac{3}{2})} I_\nu(z) \tag{10}$$

Consider the function

$$F_1 = \frac{1}{2}\left[1 + \frac{\sin 2\mu\pi}{\sin 2\nu\pi}\right] \tag{11}$$

If $(\mu - \nu)$ is a positive integer or zero, then $F_1 = 1$; if $(\mu + \nu)$ is a positive integer or zero, then $F_1 = 0$. Thus, if $(\mu \pm \nu)$ is a positive integer or zero, we can write

$$\begin{aligned} & H_{\mu,\nu}(z) = h_{\mu,\nu}(z) \\ & - \sqrt{\pi}\, \frac{\Gamma(\mu - \nu + 1)\Gamma(\mu + \nu + 1)}{2^{\mu+1}\Gamma(\mu + \frac{3}{2})} \left[I_\nu(z) + \frac{K_\nu(z)}{\pi} \sin \nu\pi \left(1 - \frac{\sin 2\mu\pi}{\sin 2\nu\pi}\right)\right] \end{aligned} \tag{12}$$

We take this equation as the general definition of $H_{\mu,\nu}(z)$.

4. The function $H_{\mu,\nu}(z)$ when $\mu \pm \nu$ is a negative integer. Since $h_{\mu,\nu}(z)$ is not defined when $\mu \pm \nu$ is a negative integer, neither is $H_{\mu,\nu}(z)$ as given by (12). However, we can show that $H_{\mu,\nu}(z)$ has a limit when $h_{\mu,\nu}(z)$ is undefined. Recall that $h_{\mu,\nu}(z)$ can be replaced by $H_{\mu,\nu}(z)$ in (4)–(6). It is easy to see that we can compute $H_{\nu-p,\nu}(z)$ in terms of $H_{\nu-1,\nu}(z)$ by repeated application of (4). Here p is a positive integer, and it is sufficient to consider the case where $(\mu - \nu)$ is a negative integer since $H_{\mu,\nu}(z)$ is an even function in ν.

To evaluate $H_{\nu-1,\nu}(z)$, we consider the limiting form of (4). We have

$$H_{\nu-1,\nu}(z) = \lim_{\mu \to \nu} \frac{(2\mu + 1)H_{\mu,\nu}(z) + e^{-z}z^{\mu}}{(\mu - \nu)(\mu + \nu)} \tag{13}$$

Both numerator and denominator are analytic functions of μ near $\mu = \nu$ and vanish when $\mu = \nu$. Hence, employing L'Hospital's theorem and (12), we obtain the result

$$H_{\nu-1,\nu} = 2^{\nu-1}\Gamma(\nu)I_{\nu}(z)\log 2z + 2^{\nu-1}\Gamma(\nu)\frac{\cos 2\nu\pi}{\cos \nu\pi}K_{\nu}(z) + 2^{\nu-1}\Gamma(\nu)\frac{e^{-z}}{\sqrt{\pi}}$$
$$\cdot\sum_{m=0}^{\infty}\frac{(2z)^{\nu+m}\Gamma(\nu + m + \tfrac{1}{2})}{m!\Gamma(2\nu + m + 1)}\{\psi(\nu + m + \tfrac{1}{2}) - \psi(m + 1) - \psi(2\nu + m + 1)\} \tag{14}$$

This formula is nugatory when ν is a negative integer or zero, and when ν is an odd multiple of $\pm\frac{1}{2}$. The situation where $\nu = -n$, n a positive integer, is easily disposed of, since here we define the function by the relation $H_{-n-1,-n}(z) = H_{-n-1,n}(z)$. When $\nu = 0$, we again employ the limiting form of (4). Apply L'Hospital's theorem, then with the aid of (12) and (9), we get

$$H_{-1,0}(z) = -\tfrac{1}{2}I_0(z)(\log 2z)^2 - K_0(z)\log 2z + \frac{e^{-z}}{2\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(2z)^m\Gamma(m + \tfrac{1}{2})}{(m!)^2}$$
$$\cdot[\psi'(m + \tfrac{1}{2}) - 2\psi'(m + 1) + \{\psi(m + \tfrac{1}{2}) - 2\psi(m + 1)\}^2] \tag{15}$$

5. The function $H_{\nu-1,\nu}(z)$ when ν is an odd multiple of $\pm 1/2$. If ν is an odd multiple of $\frac{1}{2}$ or $-\frac{1}{2}$, then $H_{\nu-1,\nu}(z)$ can be represented by $H_{n-\frac{1}{2},n+\frac{1}{2}}(z)$ or $H_{-n-\frac{1}{2},n+\frac{1}{2}}(z)$ respectively. The former cannot be defined by (3) or (7); the latter cannot be defined by (3), but (7) yields an asymptotic expansion valid for large z. Employing the recurrence relations (4) – (6), we find

$$H_{n+\frac{1}{2},n+\frac{1}{2}}(z) = -\frac{(2n + 1)^2}{z}H_{n-\frac{1}{2},n+\frac{1}{2}}(z) + (4n^2 - 1)H_{n-\frac{1}{2},n-\frac{1}{2}}(z)$$
$$+ \frac{e^{-z}z^{n-\frac{1}{2}}}{2n} - \frac{e^{-z}z^{n+\frac{1}{2}}}{2(n + 1)} \tag{16}$$

Hence, $H_{\nu-1,\nu}(z)$ is known for ν an odd multiple of $\pm\frac{1}{2}$, provided that $H_{-\frac{1}{2},\frac{1}{2}}(z)$, $H_{\frac{1}{2},\frac{1}{2}}(z)$, $H_{-\frac{1}{2},\frac{1}{2}}(z)$ and $H_{-\frac{1}{2},\frac{1}{2}}(z)$ can be computed. To evaluate the latter, we construct the solution of (2) by the method of variation of parameters. Thus

$$y = I_{\nu}(z)\int^{z} e^{-z}z^{\nu}K_{\nu}(z)\,dz - K_{\nu}(z)\int^{z} e^{-z}z^{\nu}I_{\nu}(z)\,dz \tag{17}$$

Now the descending series for both $I_\nu(z)$ and $K_\nu(z)$ is composed of a finite number of terms whenever ν is an odd multiple of $\frac{1}{2}$. If these terminating series are substituted in (17), the four functions of interest can be evaluated by quadrature. We obtain

$$H_{-\frac{1}{2},\frac{1}{2}}(z) = \frac{1}{2\sqrt{z}}\left[-e^{-z}\log z + e^{z}Ei(-2z)\right] \tag{18}$$

$$H_{\frac{1}{2},\frac{1}{2}}(z) = \frac{1}{2\sqrt{z}} \cdot \left[e^{-z}\left(z - \frac{1}{2z} + \frac{3}{2}\right) - e^{-z}\left(1 + \frac{1}{z}\right)\log z - e^{z}\left(1 - \frac{1}{z}\right)Ei(-2z)\right] \tag{19}$$

$$H_{-\frac{1}{2},\frac{1}{2}}(z) = -\frac{e^{z}}{\sqrt{z}}Ei(-2z) \tag{20}$$

$$H_{-\frac{1}{2},\frac{1}{2}}(z) = -\frac{1}{2\sqrt{z}}\left[\frac{e^{-z}}{3z} + \frac{2}{3}e^{z}\left(1 - \frac{1}{z}\right)Ei(-2z)\right] \tag{21}$$

where $Ei(-x) = \int_{\infty}^{x} (e^{-t}/t)\,dt$. This integral is well known and is extensively tabulated.[5]

As a final remark, we note that if both the numbers $\mu \pm \nu + 1$ have positive real parts, then the lower limits of the integrals in (17) may be set to zero. Thus $\mu \pm \nu$ is not a negative integer and so (17) is an integral representation of the function $h_{\mu,\nu}(z)$.

6. Evaluation of some indefinite and infinite integrals involving Bessel functions. If we let $R_\nu(z)$ represent $K_\nu(z)$ or $I_\nu(z)$, then

$$\Delta_\nu R_\nu(z) = 0 \tag{22}$$

Following McLachlan and Meyers[6], multiply (22) by $H_{\mu,\nu}(z)$ and (2) by $R_\nu(z)$ and subtract the latter from the former. Integrating the resulting expression and employing the recurrence relations (4)–(6), one obtains

$$\begin{aligned}\int^{z} e^{-z}x^{\mu}R_\nu(x)\,dx &= z[R_\nu(z)H'_{\mu,\nu}(z) - R'_\nu(z)H_{\mu,\nu}(z)] \\ &= z\left[\frac{(\mu-\nu)}{(\mu+\nu+1)}H_{\mu,\nu+1}(z)R_\nu(z) + \epsilon H_{\mu,\nu}(z)R_{\nu+1}(z) + \frac{e^{-z}z^{\mu}R_\nu(z)}{(\mu+\nu+1)}\right]\end{aligned} \tag{23}$$

where $\epsilon = +1$ or -1 according as $R_\nu(z)$ is $K_\nu(z)$ or $I_\nu(z)$, respectively. In particular, if $\mu = \nu$, then

$$\int_0^{z} e^{\pm x}x^{\nu}I_\nu(x)\,dx = \frac{e^{\pm z}z^{\nu+1}}{2\nu+1}\left[I_\nu(z) \mp I_{\nu+1}(z)\right] \tag{24}$$

$$\int_0^{z} e^{\pm x}x^{\nu}K_\nu(x)\,dx = \frac{e^{\pm z}z^{\nu+1}}{2\nu+1}\left[K_\nu(z) \pm K_{\nu+1}(z)\right] \mp \frac{2^{\nu}\Gamma(\nu+1)}{2\nu+1} \tag{25}$$

If $Q_\nu(z)$ represents any of the Bessel functions of the first three kinds, then

$$\int^z e^{-ix} x^\mu Q_\nu(x)\,dx = e^{-(\mu+1)i\pi/2} z\,[Q_\nu(z)T'_{\mu,\nu}(z) - T_{\mu,\nu}(z)Q'_\nu(z)]$$

$$= e^{-(\mu+1)i\pi/2} z\left[\frac{i(\mu-\nu)(\mu-\nu-1)}{(2\mu+1)}\,Q_\nu(z)T_{\mu-1,\nu+1}(z) + T_{\mu,\nu}(z)Q_{\nu+1}(z)\right] \quad (26)$$

$$+ \frac{e^{-iz} z^{\mu+1} Q_\nu(z)}{(2\mu+1)}$$

where $T_{\mu,\nu}(z) = H_{\mu,\nu}(iz)$. If $\mu = \nu$, then

$$\int^z e^{ix} x^\nu Q_\nu(x)\,dx = \frac{e^{iz} z^{\nu+1}}{2\nu+1}\left[Q_\nu(z) - iQ_{\nu+1}(z)\right] \quad (27)$$

It is to be noted that (23) and (26) are integrals of the type previously considered by the author[2]. In that paper, recurrence relations in μ and ν are presented for the integrals in these expressions. It is shown, for example, that if the left hand side of (23) is known for $\mu = \nu = 0$, then employment of the recursion formulae yields values of this function for all integral values of μ and ν. The formulae given here are more general and of a different character, though it is true that (23) and (26) coupled with equations (4)–(6) constitute an alternative recursive scheme for the computation of the integrals in question. The principal difference in the present analysis is that series developments, both convergent and asymptotic, are derived for direct calculation of the integrals mentioned.

For an application of the formulae of this section, some infinite integrals of interest can be evaluated. We have

$$\int_0^\infty e^{-z} z^{\mu-1} K_\nu(z)\,dz = \frac{\sqrt{\pi}\,\Gamma(\mu-\nu)\Gamma(\mu+\nu)}{2^\mu\Gamma(\mu-\tfrac{1}{2})}; \qquad R(\mu) > |R(\nu)| \quad (28)$$

$$\int_0^\infty e^{-z} z^{\mu-1} I_\nu(z)\,dz = \frac{\Gamma(\mu+\nu)\Gamma(\tfrac{1}{2}-\mu)}{\sqrt{\pi}\,2^\mu\Gamma(\nu-\mu+1)}; \quad R(\mu+\nu) > 0, \quad R(\mu) < \tfrac{1}{2} \quad (29)$$

$$\int_0^\infty e^{-iz} z^{\mu-1} J_\nu(z)\,dz = \frac{e^{-(\mu+\nu)i\pi/2}\Gamma(\mu+\nu)\Gamma(\tfrac{1}{2}-\mu)}{\sqrt{\pi}\,2^\mu\Gamma(\nu-\mu+1)}; \qquad R(\mu) < \tfrac{1}{2} \quad (30)$$

Here the letter R stands for real part. In the latter equation, the real part of the integral exists if $R(\mu+\nu) > 0$ while the imaginary part exists if $R(\mu+\nu) > -1$. If $(\mu+\nu)$ is a positive odd multiple of $\frac{1}{2}$, then the real and imaginary components differ at most only in sign. If $(\mu+\nu)$ is odd (even), the real (imaginary) part vanishes. If $(\mu+\nu)$ is zero, then

$$\int_0^\infty \frac{\sin z J_\nu(z)}{z^{\nu+1}}\,dz = \frac{\pi}{2^{\nu+1}\Gamma(\nu+1)}; \qquad R(\nu) > -\tfrac{1}{2} \quad (31)$$

This result is a special case of Struve's integral.[7] If the latter equation were valid when $\nu = -\frac{1}{2}$, we then should be able to compute $\int_0^\infty (\operatorname{Sin} z/z)\,dz$ (which

equals $\pi/2$). However, it is seen that the right hand side of (31) does not give the correct value of the integral in this case, and so this equation *qua* function of ν is not continuous at $\nu = -\frac{1}{2}$. To obtain a result which includes as a special case the infinite Sin integral, we return to (30) and put $\nu = \pm\frac{1}{2}$. Then

$$\int_0^\infty \frac{\sin z}{z^{\frac{1}{2}-\mu}}\,dz = \Gamma(\mu - \tfrac{1}{2})\sin(\mu - \tfrac{1}{2})(\pi/2); \qquad -\tfrac{1}{2} < R(\mu) < \tfrac{3}{2} \tag{32}$$

Also

$$\int_0^\infty \frac{\sin^2 z}{z^{\frac{1}{2}-\mu}}\,dz = \frac{\Gamma(\mu + \frac{1}{2})\sin(\mu + \frac{1}{2})(\pi/2)}{2^{\mu+1}(\frac{1}{2} - \mu)}; \qquad -\tfrac{3}{2} < R(\mu) < \tfrac{1}{2} \tag{33}$$

$$\int_0^\infty \frac{\cos^2 z}{z^{\frac{1}{2}-\mu}}\,dz = \frac{\Gamma(\mu - \frac{1}{2})\cos(\mu - \frac{1}{2})(\pi/2)}{2^{\mu+1}}; \qquad \tfrac{1}{2} < R(\mu) < \tfrac{3}{2} \tag{34}$$

To derive (28)–(30) it is convenient at first to assume that the series given by (7) is defined and terminates. The validity of the formulae for general μ and ν, provided the integrals exist, obtains on appeal to the theory of analytic continuation. Equation (28) is deducible from a more general result given by Watson[8]. If $R(a + ib) > 0$, $R(a - ib) > 0$ and $R(\mu + \nu) > 0$, the value of $\int_0^\infty e^{-at}J_\nu(bt)t^{\mu-1}\,dt$ is known[9]. But neither (29) nor (30) can be deduced from this result, since the conditions on $R(a \pm ib)$ are not satisfied.

7. Evaluation of an infinite integral. For a final application, we discuss the evaluation of a certain infinite integral and derive some results which are somewhat related to the main part of the paper. As a particular consequence of interest, we note that a formula of Watson is in error. Consider the function

$$G_\nu(z) = \int_0^\infty e^{-z\operatorname{Cosh} t}\sinh \nu t\,dt; \qquad R(z) > 0 \tag{35}$$

Manifestly $\Delta_\nu G_\nu(z) = \nu e^{-z}$. Now $G_0(z)$ vanishes for all z and $\lim_{\nu\to\infty} G_\nu(z) = 0$. Hence in the neighborhood of infinity, there is no complementary solution of the latter equation. The asymptotic expansion for $G_\nu(z)$ follows from (7). If $\nu \to n$, $G_n(z) = nH_{-1,n}(z)$. By L'Hospital's theorem,

$$H_{-1,0}(z) = \int_0^\infty te^{-z\operatorname{Cosh} t}\,dt; \qquad R(z) > 0 \tag{36}$$

and the latter furnishes an integral representation of the series given by (15). If

$$A_\nu(z, \theta) = \int_0^\theta e^{z\cos t}\cos \nu t\,dt \tag{37}$$

then

$$\Delta_\nu A_\nu(z, \theta) = e^{z\cos\theta}(z\sin\theta\cos\nu\theta - \nu\sin\nu\theta) \tag{38}$$

and in the neighborhood of the origin $A_\nu(z, \pi) = -\nu \sin \nu\pi\, h_{-1,\nu}(z)$. It follows from (12) that

$$\tfrac{1}{2}\pi[I_\nu(z) + I_{-\nu}(z)] = A_\nu(z, \pi) + \sin \nu\pi G_\nu(z) \tag{39}$$

This is an alternative proof of a known result[10]. Hence, the above formulae afford a convenient scheme for the computation of $G_\nu(z)$. Watson records that $G_\nu(z)$ does not appear to be expressible in a simple form[10]. For if,

$$F(m, \nu, \theta) = \int_0^\theta \cos^m t \cos \nu t\, dt \tag{40}$$

then

$$A_\nu(z, \theta) = \sum_{m=0}^{\infty} \frac{F(m, \nu, \theta)}{m!} z^m \tag{41}$$

It is stated that[10]

$$F(m, \nu, \pi) = \frac{(-)^m \sin \nu\pi}{2^m(\nu + m)}\, {}_2F_1\left(-m, \frac{\nu + m}{2};\ 1 - \frac{\nu + m}{2};\ -1\right) \tag{42}$$

and (42) coupled with (39) and (41) forms the basis for Watson's contention. Now equation (42) is not correct, and in the next paragraph we show how the more general coefficient $F(m, \nu, \theta)$ can be determined. In any event, $F(m, \nu, \pi)$ is a finite series of $(m + 1)$ terms, and to evaluate $A_\nu(z, \pi)$ and thence $G_\nu(z)$ using Watson's outline is indeed more complicated than the approach presented in this paper.

8. The computation of $F(m, \nu, \theta)$. If we apply the operator Δ_ν to (41) and combine with (38), then it is easy to establish upon equating like powers of z that

$$\{(m + 2)^2 - \nu^2\}F(m + 2, \nu, \theta) - (m + 1)(m + 2)F(m, \nu, \theta)$$
$$= (m + 2) \sin \theta \cos \nu\theta \cos^{m+1}\theta - \nu \sin \nu\theta \cos^{m+2}\theta \tag{43}$$

Since $F(0, \nu, \theta)$ and $F(1, \nu, \theta)$ are easily computed, the latter furnishes a recursion formula to compute $F(m, \nu, \theta)$. Equating the particular solution of (38) and (41) both for $\theta = \pi$, we deduce that

$$\sum_{m=0}^{\infty} \frac{F(m, \nu, \pi)}{m!} z^m = \sqrt{\pi}\, e^{-z} \sum_{m=0}^{\infty} \frac{(2z)^m \Gamma(m + \tfrac{1}{2})}{\Gamma(m + 1 + \nu)\Gamma(m + 1 - \nu)} \tag{44}$$

and so the right hand side of (44) is a generating function for the elements $F(m, \nu, \pi)$. If ν becomes an integer, then employing the commonly used expansion for $I_n(z)$, we obtain

$$F(p, n, \pi) = 0; \qquad p \neq n + 2m$$
$$F(n + 2m, n, \pi) = \frac{\pi(n + 2m)!}{2^{n+2m} m!(n + m)!} \tag{45}$$

Finally,

$$A_\nu(z, \pi/2) = -i\, s_{0,\nu}(iz)\cos \nu\pi/2 - \nu\, s_{-1,\nu}(iz)\sin \nu\pi/2 \tag{46}$$

where $s_{\mu,\nu}(iz)$ is the Lommel function in ascending powers of z, and so

$$F(m, \nu, \pi/2) = \frac{\pi}{2}\,\frac{m!}{2^m\Gamma[\frac{1}{2}(m-\nu)+1]\Gamma[\frac{1}{2}(m+\nu)+1]} \tag{47}$$

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LAGRANGIAN INTEGRATION COEFFICIENTS FOR DISTANCE FUNCTIONS TAKEN OVER RIGHT CIRCULAR CYLINDERS

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Introduction. Let dv and dv' be two volume elements, a distance r apart in a right circular cylinder of diameter a and length ca. Let $t(r)$ be an arbitrary function of r, and consider the sixfold integral

$$I = \iint t(r)\, dv\, dv'/r^2 \tag{1}$$

taken over all elements dv, dv' within the cylinder. I expect an integral of this form may arise in various studies: I came upon it in tackling a problem (for the Department of Agriculture at Oxford) on radiation damage to plant tissues; and possibly it occurs in measuring the neutron emission from the slugs of a reactor pile. In this paper however I shall merely give tables for evaluating (1) numerically as a function of c for $c = 1(1)10$ when $t(r)$ is any empirically specified function.

I have discussed the theory elsewhere [(1), (2), (3)], and shown for $c \geqq 1$

$$I \doteq \frac{\pi^2 a^4}{16}\left\{c\int_0^c t(a\xi)\,\frac{g(\xi)\,d\xi}{\xi^2} + \int_0^c t(a\xi)\,\frac{h(\xi)\,d\xi}{\xi^2} + t(ac)k(c)\right\}, \tag{2}$$

where, with the usual notation for the hypergeometric function,

$$g(\xi) = \begin{cases} 16\,\xi^3[1 - \xi F(-\frac{1}{2}, \frac{1}{2}; 2; \xi^2)], & 0 \leqq \xi \leqq 1 \\ 16\,\xi^3[1 - F(-\frac{1}{2}, \frac{1}{2}; 2; \xi^{-2})], & 1 \leqq \xi \leqq c \end{cases} \tag{3}$$

$$h(\xi) = \begin{cases} -2\xi + 4\pi^{-1}\xi[(1 - 4\xi^2)\arccos\xi + \xi(2\xi^2 + 1)(1 - \xi^2)^{\frac{1}{2}}], & 0 \leqq \xi \leqq 1 \\ -2\xi, & 1 \leqq \xi \leqq c \end{cases} \tag{4}$$

$$k(c) = \tfrac{16}{3}c^3[F(-\tfrac{3}{2}, \tfrac{1}{2}; 2; c^{-2}) - 1 + \tfrac{3}{8}c^{-2} - \tfrac{3}{64}c^{-4}]. \tag{5}$$

Provided that $t(a\xi)$ behaves fairly smoothly in the range $c \leqq \xi \leqq \sqrt{(c^2 + 1)}$, the approximation in (2) should be satisfactory. The error committed is worst when c is small: for $c = 2$, equation (2) may only be correct to about 4 or 5 significant figures, but for $c = 10$ one may expect 7 or 8 correct significant figures. It is not possible to state here precisely how accurate (2) will be without knowing the precise functional form of $t(a\xi)$. But a reader, who has in any particular instance knowledge of the behaviour of $t(a\xi)$ for $c \leqq \xi \leqq \sqrt{(c^2 + 1)}$, and who consults Table I of (3), will be able to assess the accuracy for himself quite easily.

Weighted Lagrangian integration coefficients. The numerical evaluation of definite integrals by means of an $(n + 1)$-point Lagrangian integration formula

$$\int_0^n \phi(z)\, dz \doteq \sum_{j=0}^{n} L_j^{(n)}\phi(j) \tag{6}$$

is discussed in (4). A slight modification of this procedure enables one to evaluate indefinite integrals, in which a factor of the integrand is specified and absorbed into the integration coefficients. Suppose, in fact, that we wish to evaluate

$$\int_{\alpha_0}^{c} \phi(z)\chi(z)\,dz$$

by means of $K + 1$ abutting formulae of the type (6), of which the kth ($k = 0, 1, 2, \cdots, K$) formula evaluates

$$\int_{\alpha_k}^{\alpha_{k+1}} \phi(z)\chi(z)\,dz$$

from $n_k + 1$ equidistant points z covering this interval. Thus, with h_k defined by

$$\alpha_{k+1} = \alpha_k + n_k h_k\,, \qquad \alpha_K + n_K h_K = c,$$

we get

$$\int_{\alpha_k}^{\alpha_{k+1}} \phi(z)\chi(z)\,dz = \sum_{j=0}^{n_k} h_k L_j^{(n_k)}\,\phi(\alpha_k + jh_k)\chi(\alpha_k + jh_k).$$

So we define

$$\Lambda(\alpha_k + jh_k) \doteq h_k L_j^{(n_k)}\chi(\alpha_k + jh_k);$$

and there results

$$\int_{\alpha_0}^{c} \phi(z)\chi(z)\,dz \doteq \sum_{k=0}^{K}\sum_{j=0}^{n_k} \Lambda(\alpha_k + jh_k)\phi(\alpha_k + jh_k). \tag{7}$$

There is no reason why h_k and n_k should not depend upon k, and so we may have $\Lambda(\alpha_{k+1}) \neq \Lambda(\alpha_k + n_k h_k)$ even though $\alpha_{k+1} = \alpha_k + n_k h_k$. The quantities Λ(defined in the foregoing manner with any convenient values of n_k, h_k) may be called weighted Lagrangian integration coefficients. We may tabulate $\Lambda(\alpha)$ as a function of α for $\alpha = \alpha_k + jh_k$, ($j = 0, 1, \cdots, n_k$; $k = 0, 1, 2, \cdots$), with the provision that for the special subset $\alpha = \alpha_1, \alpha_2, \cdots$ there will be two values of $\Lambda(\alpha)$, the first being $\Lambda(\alpha_k + n_k h_k)$ and the second being $\Lambda(\alpha_{k+1})$. Now we may also treat K as a current integer variable, of which $c = \alpha_K + n_K h_K$ is a function. Then we can evaluate the integral (7) as a function of c (for $K = 1, 2, 3, \cdots$) by evaluating the double sum on the right hand side of (7) as a single sum taken over all tabulated values of α up to and including the first value $c = \alpha_K + n_K h_K$ but excluding the second value $c = \alpha_{K+1}$. At the same time, if there are certain values of c for which we do not wish to tabulate the integral, it will save a multiplication if we telescope two entries together by redefining the appropriate weighted Lagrangian coefficient as $\Lambda(\alpha_{K+1}) + \Lambda(\alpha_K + n_K h_K)$.

This procedure has been followed in constructing the table at the end of this paper. The function $G(\xi)$ consists of the weighted coefficients Λ with $\xi^{-2}g(\xi)$ for χ and $\alpha_0 = 0$. Abutting 11-point Lagrangian formulae are used for $\xi = 0(0.01)2$

(0.05)3; and 5-point formulae for $\xi = 3(0.25)10$. The double entries (referred to above) have (for convenience) been telescoped into single entries except at $\xi = 1(1)10$. For these exceptional points there remain the pairs of entries, bearing the arguments 1.00_1, 1.00_2; 2.00_1, 2.00_2; $\cdots$; 10.00_1, 10.00_2. Write $\phi = t$; and then

$$\int_0^c t(a\xi)g(\xi)\,d\xi/\xi^2 \doteq \sum t(a\xi)G(\xi), \tag{8}$$

where, for $c = 1(1)10$, the summation on the right of (8) is taken over all entries for ξ in the table from $\xi = 0$ up to and including $\xi = c$ with the suffix 1. In this way a single progressive summation tabulates the left-hand side of (8) for each $c = 1(1)10$.

By writing $h(\xi)$ for $g(\xi)$ in (8) we could obtain a similar set of Lagrangian coefficients $H'(\xi)$ for evaluating

$$\int_0^c t(a\xi)\tfrac{1}{2}h(\xi)\,d\xi \doteq \sum t(a\xi)H'(\xi).$$

But we can do better than this by adding $k(c)$ to each first entry $H'(c)$, with $c = 1.00_1, 2.00_1, \cdots, 10.00_1$, and subtracting $k(c)$ from each second entry $H'(c)$, with $c = 1.00_2, 2.00_2, \cdots, 10.00_2$. If these new modified Lagrangian coefficients are called $H(\xi)$, we now have

$$\int_0^c t(a\xi)h(\xi)\,d\xi/\xi^2 + t(ac)k(c) \doteq \sum t(a\xi)H(\xi), \tag{9}$$

where the summation on the right-hand side of (9) is over the same range of values of ξ as in (8). Combining (2), (8), and (9) we get the final result

$$I \doteq (\pi^2a^4/16)\{c[\sum t(a\xi)G(\xi)] + [\sum t(a\xi)H(\xi)]\}. \tag{10}$$

The table at the end of this paper gives values of $-H(\xi)$, the minus sign being introduced to avoid the cumulation of negative products on the product register of a desk calculator.

For checking a calculation we further define and tabulate

$$J(\xi) = G(\xi) + H(\xi). \tag{11}$$

The construction of the table was checked by putting $a = 1$ and integrating the special case $t(a\xi) = \xi^2$, for which $I = \pi^2c^2/16$. The last significant figure in the table is often conjectural: for the reasons explained in (5), it is however better to retain it in this form rather than round it off.

The approximation implicit in (2) forbids the retention of more than 8 places of decimals in the table. There are occasions when the full accuracy of the table is wanted, for instance in investigating how far empirical formulae will represent the integral; and this consideration has governed the number of figures in the table and the choice of the intervals of ξ therein. The reader who can manage with a less accurate quadrature may however drop the final digits of the entries in the

table, or he can save himself some multiplications by widening the interval of ξ and recomputing $G(\xi)$ and $H(\xi)$ from the tables of $g(\xi)$ and $h(\xi)$ given in (3). His best course in any such event will depend naturally upon the number of integrands $t(a\xi)$ he has to handle.

Use of table. To calculate I for $c = 1(1)10$ and fixed a, tabulate $t(a\xi)$ for $\xi = 0(0.01)2(0.05)3(0.25)10$. Set these quantities successively on the keyboard of a desk calculator equipped with automatic multiplication*, and multiply by the corresponding entries $G(\xi)$ taken from the table. The multipliers will cumulate in the multiplier register, which may be periodically checked against the interspersed italic entries (cumulative totals of multipliers) shown in the $G(\xi)$ column. Record the results

$$I_1(c) \doteq \int_0^c t(a\xi)g(\xi)\,d\xi/\xi^2, \qquad c = 1(1)10$$

showing in the product register immediately after each multiplication for $\xi = 1.00_1, 2.00_1, \cdots, 10.00_1$.

Repeat the process with $-H$ in place of G, and so determine

$$-I_2(c) \doteq -\int_0^c t(a\xi)h(\xi)\,d\xi/\xi^2 - t(ac)k(c), \qquad c = 1(1)10.$$

Repeat once more with J in place of G, and so determine

$$I_3(c) \doteq \int_0^c t(a\xi)\{g(\xi) + h(\xi)\}\,d\xi/\xi^2 + t(ac)k(c), \qquad c = 1(1)10.$$

For each value of c, set 1 on the keyboard and multiply by $I_1(c)$. Leaving the machine uncleared, set c on the keyboard and multiply by $-[-I_2(c)]$. The product register will now show

$$16\,I/\pi^2a^4, \tag{12}$$

provided that the multiplier register checks to $I_3(c)$. Values of $I = I(c)$ for non-integral c satisfying $1 \leqq c \leqq 10$ will follow upon interpolation in $I(c)$, $c = 1(1)10$.

Acknowledgement. I am indebted to Miss M. Callow, Miss I. Strong, and Miss M. Wild, who calculated and checked the table.

* The method is particularly well suited to punched card or teletype machines. The calculation can be carried out (less conveniently) on a desk machine without automatic multiplication, provided that other arrangements are made for checking the cumulated sum of multipliers.

TABLE

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
0.00	+0.0429 3464	+0.0000 0000	+0.0429 3464
0.01	+0.2812 1693	+0.0014 0822	+0.2798 0871
0.02	−0.1270 7644	−0.0012 7469	−0.1258 0175
0.03	+0.7060 7912	+0.0106 4069	+0.6954 3843
0.04	−0.6684 0384	−0.0134 5225	−0.6549 5159
0.05	+1.0849 4008	+0.0273 3916	+1.0576 0092
0.06	−0.6544 9211	−0.0198 2404	−0.6346 6807
0.07	+0.6769 9130	+0.0239 6394	+0.6530 2736
0.08	−0.1193 0444	−0.0048 3480	−0.1144 6964
0.09	+0.2585 1824	+0.0118 0686	+0.2467 1138
0.10	+0.0772 9309	+0.0039 2939	+0.0733 6370
	+1.5586 9657	*+0.0397 0248*	*+1.5189 9409*
0.11	+0.2528 5851	+0.0141 6618	+0.2386 9233
0.12	−0.1141 3741	−0.0069 8884	−0.1071 4857
0.13	+0.6334 8556	+0.0421 0223	+0.5913 8333
0.14	−0.5990 1285	−0.0429 5701	−0.5560 5584
0.15	+0.9712 0305	+0.0747 7083	+0.8964 3222
0.16	−0.5852 0611	−0.0481 5455	−0.5370 5156
0.17	+0.6046 1733	+0.0529 7031	+0.5516 4702
0.18	−0.1064 2408	−0.0098 9298	−0.0965 3110
0.19	+0.2303 3124	+0.0226 4910	+0.2076 8214
0.20	+0.0687 8172	+0.0071 3502	+0.0616 4670
	+2.9151 9353	*+0.1455 0877*	*+2.7696 9076*
0.21	+0.2247 3610	+0.0245 3300	+0.2002 0310
0.22	−0.1013 1603	−0.0116 1300	−0.0897 0303
0.23	+0.5616 0834	+0.0674 5434	+0.4941 5400
0.24	−0.5303 6057	−0.0666 2781	−0.4637 3276
0.25	+0.8587 6585	+0.1126 5044	+0.7461 1541
0.26	−0.5167 6653	−0.0706 7271	−0.4460 9382
0.27	+0.5331 8491	+0.0759 1243	+0.4572 7248
0.28	−0.0937 2158	−0.0138 7325	−0.0798 4833
0.29	+0.2025 5617	+0.0311 3549	+0.1714 2068
0.30	+0.0604 0166	+0.0096 3024	+0.0507 7142
	+4.1142 8185	*+0.3040 3194*	*+3.8102 4991*
0.31	+0.1970 7058	+0.0325 5583	+0.1645 1475
0.32	−0.0887 1356	−0.0151 7008	−0.0735 4348
0.33	+0.4910 1840	+0.0868 3344	+0.4041 8496
0.34	−0.4629 9579	−0.0846 0269	−0.3783 9310
0.35	+0.7485 3326	+0.1412 1682	+0.6073 1644
0.36	−0.4497 2797	−0.0875 3175	−0.3621 9622
0.37	+0.4632 7720	+0.0929 5855	+0.3703 1865
0.38	−0.0813 0145	−0.0168 0701	−0.0644 9444
0.39	+0.1754 2342	+0.0373 3803	+0.1380 8539
0.40	+0.0522 2300	+0.0114 3778	+0.0407 8522

TABLE—*Cont.*

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
	+5.1590 8894	*+0.5022 6086*	*+4.6568 2808*
0.41	+0.1700 9546	+0.0383 1314	+0.1317 8232
0.42	−0.0764 3737	−0.0176 9738	−0.0587 3999
0.43	+0.4223 2303	+0.1004 5733	+0.3218 6570
0.44	−0.3975 0393	−0.0970 9827	−0.3004 0566
0.45	+0.6414 7342	+0.1608 3888	+0.4806 3454
0.46	−0.3846 8570	−0.0989 6494	−0.2857 2076
0.47	+0.3955 2206	+0.1043 6137	+0.2911 6069
0.48	−0.0692 7657	−0.0187 4087	−0.0505 3570
0.49	+0.1491 8263	+0.0413 6237	+0.1078 2026
0.50	+0.0443 2196	+0.0125 9065	+0.0317 3131
	+6.0541 0393	*+0.7276 8314*	*+5.3264 2079*
0.51	+0.1440 6547	+0.0419 1776	+0.1021 4771
0.52	−0.0646 0498	−0.0192 4808	−0.0453 5690
0.53	+0.3561 8925	+0.1086 3450	+0.2475 5475
0.54	−0.3345 3035	−0.1044 1846	−0.2301 1189
0.55	+0.5386 5743	+0.1720 3053	+0.3666 2690
0.56	−0.3223 0079	−0.1052 9513	−0.2170 0566
0.57	+0.3306 1942	+0.1104 6858	+0.2201 5084
0.58	−0.0577 7327	−0.0197 3860	−0.0380 3467
0.59	+0.1241 1431	+0.0433 5229	+0.0807 6202
0.60	+0.0367 8454	+0.0131 3360	+0.0236 5094
	+6.8053 2496	*+0.9685 2013*	*+5.8368 0483*
0.61	+0.1192 6922	+0.0435 2190	+0.0757 4732
0.62	−0.0533 5013	−0.0198 9366	−0.0334 5647
0.63	+0.2933 7953	+0.1117 7703	+0.1816 0250
0.64	−0.2748 1609	−0.1069 6890	−0.1678 4719
0.65	+0.4413 2110	+0.1754 7561	+0.2658 4549
0.66	−0.2633 3964	−0.1069 5080	−0.1563 8884
0.67	+0.2693 8485	+0.1117 4048	+0.1576 4437
0.68	−0.0469 3951	−0.0198 8447	−0.0270 5504
0.69	+0.1005 4880	+0.0434 9742	+0.0570 5138
0.70	+0.0297 1257	+0.0131 2549	+0.0165 8708
	+7.4204 9566	*+1.2139 6023*	*+6.2065 3543*
0.71	+0.0960 5048	+0.0433 2570	+0.0527 2478
0.72	−0.0428 3307	−0.0197 2796	−0.0231 0511
0.73	+0.2348 1320	+0.1104 2634	+0.1243 8686
0.74	−0.2192 6051	−0.1052 8151	−0.1139 7900
0.75	+0.3509 7436	+0.1720 7064	+0.1789 0372
0.76	−0.2087 4532	−0.1044 9360	−0.1042 5172
0.77	+0.2128 2934	+0.1087 8060	+0.1040 4874
0.78	−0.0369 6014	−0.0192 8898	−0.0176 7116
0.79	+0.0789 0210	+0.0420 4668	+0.0368 5542
0.80	+0.0232 3539	+0.0126 4373	+0.0105 9166

TABLE—*Cont.*

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
	+7.9095 0149	*+1.4544 6187*	*+6.4550 3962*
0.81	+0.0748 4977	+0.0415 9251	+0.0332 5726
0.82	−0.0332 6110	−0.0188 7470	−0.0143 8640
0.83	+0.1816 9142	+0.1052 9733	+0.0763 9409
0.84	−0.1690 5109	−0.1000 6052	−0.0689 9057
0.85	+0.2696 3341	+0.1630 0578	+0.1066 2763
0.86	−0.1597 9219	−0.0986 7155	−0.0611 2064
0.87	+0.1623 3653	+0.1023 9559	+0.0599 4094
0.88	−0.0280 9160	−0.0181 0045	−0.0099 9115
0.89	+0.0597 5991	+0.0393 3570	+0.0204 2421
0.90	+0.0175 3800	+0.0117 9320	+0.0057 4480
	+8.2851 1455	*+1.6821 7476*	*+6.6029 3979*
0.91	+0.0563 0814	+0.0386 8137	+0.0176 2677
0.92	−0.0249 4160	−0.0175 0365	−0.0074 3795
0.93	+0.1358 3213	+0.0973 7851	+0.0384 5362
0.94	−0.1260 2639	−0.0922 8794	−0.0337 3845
0.95	+0.2004 9980	+0.1499 5675	+0.0505 4305
0.96	−0.1185 6242	−0.0905 4953	−0.0280 1289
0.97	+0.1202 4209	+0.0937 4857	+0.0264 9352
0.98	−0.0207 8361	−0.0165 3597	−0.0042 4764
0.99	+0.0441 9768	+0.0358 6448	+0.0083 3320
1.00_1	+0.0064 9059	−0.0329 5899	+0.0394 4958
	+8.5583 7096	*+1.8479 6836*	*+6.7104 0260*
1.00_2	+0.0064 9059	+0.0436 9265	−0.0372 0206
1.01	+0.0417 8612	+0.0351 5563	+0.0066 3049
1.02	−0.0185 8101	−0.0158 9090	−0.0026 9011
1.03	+0.1016 8443	+0.0883 3909	+0.0133 4534
1.04	−0.0948 7818	−0.0836 8368	−0.0111 9450
1.05	+0.1518 9835	+0.1359 5517	+0.0159 4318
1.06	−0.0904 3582	−0.0821 0474	−0.0083 3108
1.07	+0.0923 7569	+0.0850 3669	+0.0073 3900
1.08	−0.0160 8438	−0.0150 0807	−0.0010 7631
1.09	+0.0344 5359	+0.0325 7540	+0.0018 7819
1.10	+0.0101 8806	+0.0097 5787	+0.0004 3019
	+8.7772 6840	*+2.0817 9347*	*+6.6954 7493*
1.11	+0.0329 7909	+0.0319 8846	+0.0009 9063
1.12	−0.0147 3653	−0.0144 7207	−0.0002 6446
1.13	+0.0810 0259	+0.0805 2147	+0.0004 8112
1.14	−0.0758 8859	−0.0763 4300	+0.0004 5441
1.15	+0.1219 5696	+0.1241 3298	−0.0021 7602
1.16	−0.0728 6772	−0.0750 2675	+0.0021 5903

TABLE—*Cont.*

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
1.17	+0.0746 8032	+0.0777 6860	−0.0030 8828
1.18	−0.0130 4460	−0.0137 3620	+0.0006 9160
1.19	+0.0280 2666	+0.0298 3797	−0.0018 1131
1.20	+0.0083 1149	+0.0089 4472	−0.0006 3323
	+8.9476 8807	*+2.2554 0965*	*+6.6922 7842*
1.21	+0.0269 7884	+0.0293 4478	−0.0023 6594
1.22	−0.0120 8726	−0.0132 8583	+0.0011 9857
1.23	+0.0666 0932	+0.0739 7501	−0.0073 6569
1.24	−0.0625 5688	−0.0701 8631	+0.0076 2943
1.25	+0.1007 6971	+0.1142 0234	−0.0134 3263
1.26	−0.0603 4598	−0.0690 7224	+0.0087 2626
1.27	+0.0619 8357	+0.0716 4508	−0.0096 6151
1.28	−0.0108 4995	−0.0126 6306	+0.0018 1311
1.29	+0.0233 5968	+0.0275 2495	−0.0041 6527
1.30	+0.0069 4139	+0.0082 5666	−0.0013 1527
	+9.0884 9051	*+2.4151 5105*	*+6.6733 3948*
1.31	+0.0225 7549	+0.0271 0472	−0.0045 2923
1.32	−0.0101 3362	−0.0122 7933	+0.0021 4571
1.33	+0.0559 4640	+0.0684 1298	−0.0124 6658
1.34	−0.0526 3704	−0.0649 4853	+0.0123 1149
1.35	+0.0849 3864	+0.1057 4291	−0.0208 0427
1.36	−0.0509 5227	−0.0639 9340	+0.0130 4113
1.37	+0.0524 2196	+0.0664 1552	−0.0139 9356
1.38	−0.0091 9113	−0.0117 4544	+0.0025 5431
1.39	+0.0198 1962	+0.0255 4474	−0.0057 2512
1.40	+0.0058 9856	+0.0076 6690	−0.0017 6834
	+9.2071 7712	*+2.5630 7210*	*+6.6441 0502*
1.41	+0.0192 1293	+0.0251 8240	−0.0059 6947
1.42	−0.0086 3700	−0.0114 1459	+0.0027 7759
1.43	+0.0477 5272	+0.0636 2885	−0.0158 7613
1.44	−0.0449, 9163	−0.0604 3821	+0.0154 4658
1.45	+0.0727 0212	+0.0984 5029	−0.0257 4817
1.46	−0.0436 7111	−0.0596 1029	+0.0159 3918
1.47	+0.0449 9052	+0.0618 9745	−0.0169 0693
1.48	−0.0078 9845	−0.0109 5183	+0.0030 5338
1.49	+0.0170 5386	+0.0238 3033	−0.0067 7647
1.50	+0.0050 8180	+0.0071 5577	−0.0020 7397
	+9.3087 7288	*+2.7008 0227*	*+6.6079 7061*
1.51	+0.0165 7287	+0.0235 1469	−0.0069 4182
1.52	−0.0074 5916	−0.0106 6363	+0.0032 0447
1.53	+0.0412 8943	+0.0594 7010	−0.0181 8067
1.54	−0.0389 4723	−0.0565 1365	+0.0175 6642

TABLE—*Cont.*

t	$G(t)$	$-H(t)$	$J(t) = G(t) + H(t)$
1.55	+0.0630 0672	+0.0920 9866	−0.0290 9194
1.56	−0.0378 8961	−0.0557 8912	+0.0178 9951
1.57	+0.0390 7730	+0.0579 5494	−0.0188 7764
1.58	−0.0068 6775	−0.0102 5868	+0.0033 9093
1.59	+0.0148 4420	+0.0223 3156	−0.0074 8736
1.60	+0.0044 2797	+0.0067 0854	−0.0022 8057
	+9.3968 2762	*+2.8296 5568*	*+6.5671 7194*
1.61	+0.0144 5544	+0.0220 5415	−0.0075 9871
1.62	−0.0065 1271	−0.0100 0538	+0.0034 9267
1.63	+0.0360 8629	+0.0558 2163	−0.0197 3534
1.64	−0.0340 7252	−0.0530 6770	+0.0189 9518
1.65	+0.0551 7370	+0.0865 1692	−0.0313 4322
1.66	−0.0332 1058	−0.0524 2833	+0.0192 1775
1.67	+0.0342 8352	+0.0544 8459	−0.0202 0107
1.68	−0.0060 3079	−0.0096 4804	+0.0036 1725
1.69	+0.0130 4693	+0.0210 1017	−0.0079 6324
1.70	+0.0038 9532	+0.0063 1392	−0.0024 1860
	+9.4739 4222	*+2.9507 0761*	*+6.5232 3461*
1.71	+0.0127 2769	+0.0207 6444	−0.0080 3675
1.72	−0.0057 3925	−0.0094 2367	+0.0036 8442
1.73	+0.0318 2766	+0.0525 9495	−0.0207 6729
1.74	−0.0300 7674	−0.0500 1783	+0.0199 4109
1.75	+0.0487 4357	+0.0815 7310	−0.0328 2953
1.76	−0.0293 6400	−0.0494 4945	+0.0200 8545
1.77	+0.0303 3701	+0.0514 0636	−0.0210 6935
1.78	−0.0053 4078	−0.0091 0602	+0.0037 6524
1.79	+0.0115 6321	+0.0198 3642	−0.0082 7321
1.80	+0.0034 5499	+0.0059 6314	−0.0025 0815
	+9.5420 7558	*+3.0648 4905*	*+6.4772 2653*
1.81	+0.0112 9753	+0.0196 1723	−0.0083 1970
1.82	−0.0050 9817	−0.0089 0589	+0.0038 0772
1.83	+0.0282 9339	+0.0497 2091	−0.0214 2752
1.84	−0.0267 5639	−0.0472 9947	+0.0205 4308
1.85	+0.0433 9369	+0.0771 6374	−0.0337 7005
1.86	−0.0261 5971	−0.0467 9087	+0.0206 3116
1.87	+0.0270 4549	+0.0486 5736	−0.0216 1187
1.88	−0.0047 6461	−0.0086 2166	+0.0038 5705
1.89	+0.0103 2280	+0.0187 8687	−0.0084 6407
1.90	+0.0030 8645	+0.0056 4929	−0.0025 6284
	+9.6027 3605	*+3.1728 2656*	*+6.4299 0949*
1.91	+0.0100 9914	+0.0185 9015	−0.0084 9101
1.92	−0.0045 6038	−0.0084 4204	+0.0038 8166

TABLE—*Cont.*

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
1.93	+0.0253 2524	+0.0471 4469	−0.0218 1945
1.94	−0.0239 6485	−0.0448 6135	+0.0208 9650
1.95	+0.0388 9098	+0.0732 0663	−0.0343 1565
1.96	−0.0234 5995	−0.0444 0358	+0.0209 4363
1.97	+0.0242 6931	+0.0461 8744	−0.0219 1813
1.98	−0.0042 7814	−0.0081 8622	+0.0039 0808
1.99	+0.0092 7444	+0.0178 4281	−0.0085 6837
2.00_1	+0.0013 8732	+0.0009 3766	+0.0004 4966
	+9.6557 1916	*+3.2708 4275*	*+6.3848 7641*
2.00_2	+0.0069 3662	+0.0151 6283	−0.0082 2621
2.05	+0.0436 0645	+0.0866 0290	−0.0429 9645
2.10	−0.0189 3920	−0.0385 9218	+0.0196 5298
2.15	+0.1012 8058	+0.2116 0292	−0.1103 2234
2.20	−0.0923 9524	−0.1977 9778	+0.1054 0254
2.25	+0.1447 0744	+0.3172 2872	−0.1725 2128
2.30	−0.0843 2909	−0.1891 9788	+0.1048 6879
2.35	+0.0843 5992	+0.1935 9417	−0.1092 3425
2.40	−0.0143 9336	−0.0337 6815	+0.0193 7479
2.45	+0.0302 2756	+0.0724 6365	−0.0422 3609
2.50	+0.0087 6791	+0.0214 6732	−0.0126 9941
	+9.8655 4875	*+3.7296 0927*	*+6.1359 3948*
2.55	+0.0278 5464	+0.0696 2194	−0.0417 6730
2.60	−0.0122 2135	−0.0311 7060	+0.0189 4925
2.65	+0.0659 9202	+0.1716 7784	−0.1056 8582
2.70	−0.0607 6230	−0.1611 6856	+0.1004 0626
2.75	+0.0960 1061	+0.2595 5077	−0.1635 4016
2.80	−0.0564 2682	−0.1554 1254	+0.0989 8572
2.85	+0.0569 0766	+0.1596 3028	−0.1027 2262
2.90	−0.0097 8542	−0.0279 4606	+0.0181 6064
2.95	+0.0207 0468	+0.0601 8168	−0.0394 7700
3.00_1	+0.0030 2449	+0.0086 1342	−0.0055 8893
	+9.9968 4696	*+4.0831 8744*	*+5.9136 5952*
3.00_2	+0.0175 3277	+0.0521 8314	−0.0346 5037
3.25	+0.0681 4546	+0.2188 0342	−0.1506 5796
3.50	+0.0219 9672	+0.0761 9048	−0.0541 9376
3.75	+0.0510 2771	+0.1896 2963	−0.1386 0192
4.00_1	+0.0097 9970	+0.0387 8545	−0.0289 8575
	+10.1653 4932	*+4.6587 7956*	*+5.5065 6976*
4.00_2	+0.0097 9970	+0.0389 9232	−0.0291 9262
4.25	+0.0396 4676	+0.1673 2026	−0.1276 7350
4.50	+0.0132 5130	+0.0592 5926	−0.0460 0796
4.75	+0.0316 9447	+0.1497 0760	−0.1180 1313
5.00_1	+0.0062 5373	+0.0310 6900	−0.0248 1527

TABLE—*Cont.*

ξ	$G(\xi)$	$-H(\xi)$	$J(\xi) = G(\xi) + H(\xi)$
	+10.2659 9528	*+5.1051 2800*	*+5.1608 6728*
5.00_2	+0.0062 5373	+0.0311 5322	−0.0248 9949
5.25	+0.0259 1830	+0.1354 4974	−0.1095 3144
5.50	+0.0088 5224	+0.0484 8485	−0.0396 3261
5.75	+0.0215 9018	+0.1236 7150	−0.1020 8132
6.00_1	+0.0043 3612	+0.0259 0568	−0.0215 6956
	+10.3329 4585	*+5.4697 9299*	*+4.8631 5286*
6.00_2	+0.0043 3612	+0.0259 4617	−0.0216 1005
6.25	+0.0182 6317	+0.1137 7778	−0.0955 1461
6.50	+0.0063 3045	+0.0410 2564	−0.0346 9519
6.75	+0.0156 5049	+0.1053 4979	−0.0896 9930
7.00_1	+0.0031 8275	+0.0222 1132	−0.0190 2857
	+10.3807 0883	*+5.7781 0369*	*+4.6026 0514*
7.00_2	+0.0031 8275	+0.0222 3313	−0.0190 5038
7.25	+0.0135 6123	+0.0980 8429	−0.0845 2306
7.50	+0.0047 5133	+0.0355 5556	−0.0308 0423
7.75	+0.0118 6429	+0.0917 5627	−0.0798 9198
8.00_1	+0.0024 3533	+0.0194 3806	−0.0170 0273
	+10.4165 0376	*+6.0451 7100*	*+4.3713 3276*
8.00_2	+0.0024 3533	+0.0194 5083	−0.0170 1550
8.25	+0.0104 6719	+0.0861 9529	−0.0757 2810
8.50	+0.0036 9730	+0.0313 7255	−0.0276 7525
8.75	+0.0093 0321	+0.0812 6984	−0.0719 6663
9.00_1	+0.0019 2341	+0.0172 7997	−0.0153 5656
	+10.4443 3020	*+6.2807 3948*	*+4.1635 9072*
9.00_2	+0.0019 2341	+0.0172 8793	−0.0153 6452
9.25	+0.0083 2320	+0.0768 7688	−0.0685 5368
9.50	+0.0029 5886	+0.0280 7018	−0.0251 1132
9.75	+0.0074 9033	+0.0729 3447	−0.0654 4414
10.00_1	+0.0015 5751	+0.0155 5294	−0.0139 9543
	+10.4665 8351	*+6.4914 6188*	*+3.9751 2163*

(Entries in italics are cumulative totals.)

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LAWS OF ATTRACTION HAVING A CERTAIN GENERALIZED NEWTONIAN PROPERTY

By H. D. Block

The Newtonian law of attraction has the property that two uniform spheres attract each other as if each had its mass concentrated at its center. The question naturally arises as to whether there are other central force laws possessing this property. We restrict ourselves here to forces which are derivable from a potential $v(r)$; that is, the force between two point masses of mass m_1, m_2 a distance r apart is $-\nabla m_1 m_2 v(r)$, and the potential energy of interaction is $w(r) = m_1 m_2 v(r)$. We further assume that $v(r)$ has two continuous derivatives outside of the origin. We then seek all such functions, $v(r)$, [and functions $\phi(a, b)$] having the property that any two uniform spheres, of masses m_1, m_2 and radii a, b respectively, placed so that the distance between their centers is R $(R > a + b)$, have a potential energy of interaction:

$$w(R, a, b) = m_1 m_2 \phi(a, b) v(R) \tag{1}$$

Bouwkamp has shown[1] that if $v(r)$ is any one of the functions $e^{-\lambda r}/r$, $\cos \lambda r/r$, $\sin \lambda r/r$ then $v(r)$ satisfies (1). Sneddon and Thornhill[2] showed that these are essentially the only solutions, provided that one assumes that $\int_{R-\eta}^{R+\eta} rv(r)dr = Rv(R)f(\eta)$. We shall show here that certain linear combinations of the solutions given by Bouwkamp are the only solutions, without making the ad hoc assumptions of Sneddon and Thornhill.

For a unit point mass at distance R from the center of a uniform sphere of density ρ and radius a the energy of interaction is[3]

$$w_1(R, a) = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \eta^2 \sin\theta \rho v(r)\, d\eta\, d\theta\, d\phi$$

$$= (2\pi/R) \int_0^{a} \int_{R-\eta}^{R+\eta} tv(t)\, dt\eta\rho\, d\eta; \text{ where } r = \sqrt{\eta^2 + R^2 - 2\eta R \cos\theta}.$$

Interchanging the order of integration and performing one integration, we have

$$w_1(R, a) = (\pi\rho/R) \int_{R-a}^{R+a} tv(t)[a^2 - (t - R)^2]\, dt, \quad \text{or,} \quad \text{letting } t = R + ax$$

and $m_1 = 4/3\pi\rho a^3$, we have

$$w_1(R, a) = (3m_1/4R) \int_{-1}^{1} (R + ax) \cdot v(R + ax) \cdot (1 - x^2)\, dx \tag{2}$$

[1] C. J. Bouwkamp, Physica. 13,501, 1947.

[2] I. N. Sneddon and C. K. Thornhill, Proc. Cambridge Phil. Soc. 45,318, 1949.

[3] See e.g. I. N. Sneddon, Fourier Transforms. McGraw Hill Co. 1951. p. 333; or the paper of Bouwkamp cited above.

If we now consider a second sphere, of radius b and density ρ', then the total energy of interaction between the two spheres is given by

$$w(R, a, b) = \int_0^{2\pi} \int_0^{\pi} \int_0^{b} r^2 \sin\theta \rho' w_1(z, a)\, dr\, d\theta\, d\phi,$$

where $z = \sqrt{R^2 + r^2 - 2rR\cos\theta}$.

Then, just as in the computation leading to (2), we have $w(R, a, b) = (3m_2/4R) \int_{-1}^{1} (R + by) \cdot w_1(R + by, a) \cdot (1 - y^2) dy$. Using the value of w_1 already computed, we have

$$w(R, a, b) = (9m_1m_2/16R) \int_{-1}^{1} \int_{-1}^{1} (1 - y^2) \cdot (1 - x^2) \cdot (R + by + ax) \cdot v(R + by + ax)\, dx\, dy \tag{3}$$

Equation (3) gives the interaction energy of two uniform spheres for any force law having a potential $v(r)$.

We now compare equation (1) with equation (3). Since $\int_{-1}^{1} \int_{-1}^{1} (1 - x^2)(1 - y^2)\, dx\, dy = 16/9$, an obvious solution is $v(R) = 1/R$, $\phi(a, b) = 1$; i.e., the Newtonian force law. However more general solutions exist. Let $Rv(R) = V(R)$, so that we require

$$\tfrac{9}{16} \int_{-1}^{1} \int_{-1}^{1} (1 - x^2) \cdot (1 - y^2) V(R + ax + by)\, dx\, dy = \phi(a, b) V(R) \tag{4}$$

Differentiating both sides twice with respect to a, [the smoothness of V under the integral implies the same for $\phi(a, b)$], we have

$$\tfrac{9}{16} \int_{-1}^{1} \int_{-1}^{1} (1 - x^2)(1 - y^2) t^2 V''(R + ax + by)\, dx\, dy = (\partial^2 \phi / \partial a^2) V(R).$$

Let $a \to 0$, $b \to 0$ and $5(\partial^2\phi/\partial a^2) \mid_{0,0} = \lambda$. Then we have $V'' = \lambda V$. This is possible for $\lambda \neq 0$ only if $V(R) = C_1 e^{-\sqrt{\lambda} R} + C_2 e^{\sqrt{\lambda} R}$, i.e. $v(R) = C_1 e^{-\sqrt{\lambda} R}/R + C_2 e^{\sqrt{\lambda} R}/R$, where C_1 and C_2 are arbitrary constants. Since λ may be either positive or negative, these are the necessary slight generalizations of the solutions given by Bouwkamp. For $\lambda = 0$ one finds the Newtonian potential plus an additive constant.

All possible potentials, $v(R)$, having now been found, the corresponding functions $\phi(a, b)$ are immediately determined by substitution of $V(R)$ in (4). It is then easily verified that these solutions actually satisfy (1).

The results may be summarized, in terms of real quantities, as follows:

THEOREM. *The only force laws which are derivable from a potential* $v(r)$ [$v(r) \in C^2$ *for* $r > 0$] *and have the property that the interaction energy,* w, *of any two uniform spheres is given by* (1) *are those derived from one of the potentials*

I: $v(r) = C_1 e^{-\beta r}/r + C_2 e^{\beta r}/r$

II: $v(r) = C_1/r + C_2$

III: $v(r) = C_1(\sin \beta r)/r + C_2(\cos \beta r)/r$,

where $\beta > 0$, C_1, C_2 are arbitrary real constants. [Note that, for $C_1 = -1, C_2 = 0$, case I gives the Yukawa potential of meson theory while case II gives the Newton Potential. If we demand further that $-v(r) \geqq 0$ and $\lim_{r\to\infty} v(r) = 0$ then these are the only solutions.]

The corresponding functions $\phi(a, b)$ are $g(a)\cdot g(b)$, where for case

$$\text{I:} \quad g(a) = (3/\beta^2 a^2)\,[\cosh \beta a - (\sinh \beta a)/\beta a], \qquad a \neq 0$$
$$= 1, \qquad a = 0;$$
$$\text{II:} \quad g(a) \equiv 1;$$
$$\text{III:} \quad g(a) = (3/\beta^2 a^2)\,[(\sin \beta a)/\beta a - \cos \beta a], \qquad a \neq 0$$
$$= 1, \qquad a = 0.$$

Now let us consider spheres whose density depends only on the distance from the center of the sphere. We call such a sphere spherically uniform. It is then readily verified that the functions $v(r)$ still satisfy equation (1) with the modification that $\phi(a, b)$ may vary with the distributions $\rho(r)$, $\rho'(r)$. This is the sense in which the writers cited above considered the problem. For example; for $v(r) = e^{\beta r}/r$ a simple computation shows that

$$\phi(a, b) = \frac{4\pi^2 \int_0^a (e^{\beta x} - e^{-\beta x}) x\rho(x)\, dx \int_0^b (e^{\beta y} - e^{-\beta y}) y\rho'(y)\, dy}{\beta^2 m_1 m_2} \tag{4}$$

If $\rho(x)$, $\rho'(y)$ are constants this reduces to the result given in the theorem, but if $\rho(x) = 1/x$, $\rho'(y) = 1/y$, then $\phi(a, b) = g(a)\cdot g(b)$ with $g(a) = (e^{\beta a} + e^{-\beta a} - 2)/\beta^{3/2} a^2$, which is a different function of a.

If we consider the problem of finding solutions, $v(r)$, to (1) which shall hold for all spherically uniform spheres and where $\phi(a, b)$ may change with the density functions $\rho(r)$, $\rho'(r)$ we shall see that the theorem still holds with the word "spherically" inserted before "uniform" and the functions $\phi(a, b)$ replaced by expression (4) for case I and

$$\phi(a, b) = (16\pi^2/m_1 m_2 \beta^2) \int_0^a y \sin \beta y \rho(y)\, dy \int_0^b z \sin \beta z \rho'(z)\, dz,$$

for case III. This follows readily from the fact that if (1) holds for all spherically uniform spheres then it holds for all uniform ones. But for these, ϕ can depend only on a and b and hence by the theorem the $v(r)$ given there are the only solutions. The replacement for $\phi(a, b)$ is easily found from the formula

$$w(R, a, b) = (4\pi^2/R) \int_0^b \int_{R-z}^{R+z} \int_0^a \int_{u-y}^{u+y} xv(x)\, dx y\rho(y)\, dy\, du z\rho'(z)\, dz,$$

which gives the potential energy of interaction for spherically uniform spheres and which may be verified in the same manner as equation (3).

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FORMULAS FOR NUMERICAL DIFFERENTIATION IN THE COMPLEX PLANE

By Herbert E. Salzer

Often one has an analytic function $f(z)$ which is tabulated in the complex plane over a Cartesian grid ($z = x + iy$), from which it is required to find its derivatives with respect to z at points $z_j = z_0 + jh$, where h is the length of the square in the grid, and j is a small complex integer. By considering the real and imaginary parts of $f(z)$ separately, namely $U(x, y)$ and $V(x, y)$, on a line parallel to either the x- or y-axis, one can apply the well-known formulas for numerical differentiation of a real function, in order to obtain either $\partial^m U/\partial x^m$ and $\partial^m V/\partial x^m$, or $\partial^m U/\partial y^m$ and $\partial^m V/\partial y^m$. Either of those pairs of real derivatives determines the m^{th} derivative from the analyticity of $f(z)$. But this procedure for complex numerical differentiation, although it involves the least amount of labor, does not offer the advantage of the closest possible association of points z_j upon which to base the approximating polynomials for $f(z)$, from which the derivatives may be obtained. The formulas for differentiation that are given here are based upon the approximation of $f(z)$ by the Lagrange polynomial of degree $n - 1$ which assumes the n values $f(z_k)$ at certain fixed points z_k. Those points z_k are chosen to lie much closer together than if they had been along a straight line at interval h. The following configurations were used for the various n-point formulas:

Lagrange's n-point polynomial approximation to $f(z)$ is

$$f(z) = \sum_k P_k(z)\, f(z_k), \tag{1}$$

where $P_k(z)$ is the well-known polynomial of degree $n - 1$ such that $P_k(z_j) = \delta_k^j$, and where the summation is over the n fixed points z_k. In all formulas given here, in order to avoid too much notation, the index n will be understood. Thus, an expression such as $P_k(z)$ will vary with n. The ν^{th} derivative of $f(z)$ can be approximated by differentiation of (1), resulting in

$$f^{(\nu)}(z) = \sum_k P_k^{(\nu)}(z) f(z_k), \tag{2}$$

* In (1), (2), (3), (3′), as well as in all tabulated formulas, the = sign denotes approximate equality, exact only for a polynomial of degree $n - 1$.

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or if $z = z_0 + Ph$,† $P_k(z) = L_k(P)$, then $P_k^{(\nu)}(z) = L_k^{(\nu)}(P)/h^\nu$, and, employing the above $z_j = z_0 + jh$, $P_k^{(\nu)}(z_j) = L_k^{(\nu)}(j)/h^\nu$. Then formula (2) becomes, for $z = z_j$,

$$h^\nu f^{(\nu)}(z_j) = \sum_k L_k^{(\nu)}(j) f(z_k). \tag{3}$$

The formulas below employ the notation $f(z_k) \equiv f_k$, $f^{(\nu)}(z_j) \equiv f_j^{(\nu)}$, and the quantities $L_k^{(\nu)}(j)$ are expressed in the form $M_k^{(\nu)}(j)/M(\nu)$, where both $M_k^{(\nu)}(j)$ and $M(\nu)$ are integers. The $M(\nu)$ is the least common denominator of $L_k^{(\nu)}(j)$ for all values of k and j. Thus (3) is expressed in the formulas below in the form

$$M(\nu)h^\nu f_j^{(\nu)} = \sum_k M_k^{(\nu)}(j) f_k\,. \tag{3'}$$

For each of the above n-point configurations, the exact integral quantities $M(\nu)$ and $M_k^{(\nu)}(j)$ are given, for $\nu = 1, 2, \cdots, n-1$, for obtaining all the derivatives at each of the points z_j of the configuration. For $\nu \geqq n$, $f^{(\nu)}(z) = 0$ when $f(z)$ is approximated by the n-point Lagrange polynomial.

The computation was performed with the aid of two convenient formulas for the coefficients $L_k^{(\nu)}(j)$, namely, the recursion formula

$$L_k^{(\nu)}(j) = [\nu/(j-k)]\,[(A_k/\nu)L^{(\nu)}(j) - L_k^{(\nu-1)}(j)], \quad \text{for } k \neq j, \tag{4}$$

and

$$L_k^{(\nu)}(j) = [A_j/(\nu+1)]\,L^{(\nu+1)}(j), \quad \text{for } k = j. \tag{5}$$

In (4) and (5), $L^{(\nu)}(j)$ denotes $[d^\nu L(P)/dP^\nu]_{P=j}$ where $L(P) \equiv \prod_j (P-j)$, the product $\prod_j$ being taken over all points of the configuration, and A_k denotes $1/\prod_j' (k-j)$ where in $\prod_j'$ the factors $(k-j)$ are taken for all points of the configuration except $j = k$. Also, a considerable amount of work was saved on the six-, eight-, and nine-point configuration by the identity (which holds only for configurations that are symmetric with respect to the 45° ray)

$$L_{b+ia}^{(\nu)}(d+ic) = i^{-\nu}\,\overline{L_{a+ib}^{(\nu)}(c+id)}. \tag{6}$$

Formulas (4) and (5) are obtained by ν-fold differentiation of

$$A_k L(P) = (P-k)L_k(P),$$

by Leibnitz's theorem. Formula (6) follows immediately from the relation $L_{b+ia}(d+ic) = \overline{L_{a+ib}(c+id)}$, for all configurations that are symmetric about the 45° ray, proof of which is furnished in the author's "Formulas for Numerical Integration of First and Second Order Differential Equations in the Complex Plane," *Jour. of Math. & Phys.*, XXIX, no. 3, 1950, pp. 207–216. For then, $L_{b+ia}^{(\nu)}(i\bar{P})$ is had by ν-fold differentiation of $\overline{L_{a+ib}(P)}$ with respect to $i\bar{P}$, when one obtains $d^\nu\,\overline{L_{a+ib}(P)}/i^\nu d\bar{P}^\nu$, which is the same as $\overline{L_{a+ib}^{(\nu)}(P)}/i^\nu$.

The calculation proceeded in the following manner: First A_k and $L^{(\nu)}(j)/\nu$ were obtained, the latter from the polynomials $L^{(\nu)}(P)/\nu$ by setting P equal to

† This argument P bears no relation to either $P_k(z)$ or to the later used $P(z)$.

each value j of the configuration $z_j = z_0 + jh$. Also, to facilitate the use of (4), an auxiliary table of $\nu/(j-k)$ was prepared for every required ν, j, and k. Then the recursion formula (4) for $k \neq j$, and formula (5) for $k = j$, were employed to calculate $L_k^{(\nu)}(j)$. It is worth noting that without the use of formulas (4) and (5), the labor in computing $L_k^{(\nu)}(j)$ would have been very much greater, because it would have necessitated, for each n-point formula, the evaluation of the complex polynomial expressions $L_k^{(\nu)}(P)$ for all values of $\nu < n$, for every k, and for P equal to every j associated with the configuration of the points z_j (or z_k). A first preliminary check upon $L_k^{(\nu)}(j)$ was performed by making use of the relation $\sum_k L_k^{(\nu)}(j) = 0$, for every ν and j. Then after $M(\nu)$ and $M_k^{(\nu)}(j)$ were obtained, a second preliminary check was performed, to assure the correct conversion from $L_k^{(\nu)}(j)$ to $M_k^{(\nu)}(j)$, employing the relation $\sum_k M_k^{(\nu)}(j) = 0$, for every ν and j. The symmetry relation (6) was used in place of calculation, wherever it held in the six-, eight-, and nine-point cases, with the exception of $j = 0$, $j = 1 + i$, or $j = 2 + 2i$, for even values of ν. The reason for this exception is that for those special values of j and ν, the symmetry rule would be applied where j is the same for both members of (6), and where either the real or imaginary parts of the two members of (6) would have opposite signs. Thus an error in such a place would not be caught in the preliminary summation check, (although it would show up in the summation check on $L_k^{(\nu+1)}(j)$, or in the final check). A functional check upon the final manuscript was performed by applying all the formulas below, at each point z_j of every n-point configuration, to calculate all the m^{th} derivatives of z^{n-1}, $m = 1, 2, \cdots, n-1$, and comparing the results with the m^{th} derivatives that were precomputed from the formula $(n-1)(n-2)\cdots(n-m)z^{n-m-1}$.

FORMULAS FOR NUMERICAL DIFFERENTIATION

Three-Point

First Derivative

$$2hf_0^{(1)} = (-2+2i)f_0 + (1-i)f_1 + (1-i)f_i$$
$$2hf_1^{(1)} = (-2-2i)f_0 + (3+i)f_1 + (-1+i)f_i$$
$$2hf_i^{(1)} = (2+2i)f_0 + (-1+i)f_1 + (-1-3i)f_i$$

Second Derivative

$$h^2 f_{\text{any}}^{(2)} = -2if_0 + (1+i)f_1 + (-1+i)f_i$$

Four-Point

First Derivative

$$2hf_0^{(1)} = (-3+3i)f_0 - 2if_1 + 2f_i + (1-i)f_{1+i}$$
$$2hf_1^{(1)} = -2if_0 + (3+3i)f_1 + (-1-i)f_i - 2f_{1+i}$$
$$2hf_i^{(1)} = 2f_0 + (1+i)f_1 + (-3-3i)f_i + 2if_{1+i}$$
$$2hf_{1+i}^{(1)} = (-1+i)f_0 - 2f_1 + 2if_i + (3-3i)f_{1+i}$$

Second Derivative

$$h^2f_0^{(2)} = -4if_0 + (3+i)f_1 + (-3+i)f_i + 2if_{1+i}$$
$$h^2f_1^{(2)} = (3-i)f_0 + 4if_1 - 2if_i + (-3-i)f_{1+i}$$
$$h^2f_i^{(2)} = (-3-i)f_0 - 2if_1 + 4if_i + (3-i)f_{1+i}$$
$$h^2f_{1+i}^{(2)} = 2if_0 + (-3+i)f_1 + (3+i)f_i - 4if_{1+i}$$

Third Derivative

$$h^3f_{\text{any}}^{(3)} = (3+3i)f_0 + (-3+3i)f_1 + (3-3i)f_i + (-3-3i)f_{1+i}$$

Five-Point

First Derivative

$$10hf_0^{(1)} = (-20+15i)f_0 - 20if_1 + (2+i)f_2 + (8+4i)f_i + 10f_{1+i}$$
$$10hf_1^{(1)} = -5if_0 + (5+15i)f_1 + (1-2i)f_2 + (-1-3i)f_i + (-5-5i)f_{1+i}$$
$$10hf_2^{(1)} = (-10+5i)f_0 + (-20-40i)f_1 + (24+7i)f_2 + (-4+8i)f_i + (10+20i)f_{1+i}$$
$$10hf_i^{(1)} = (10-5i)f_0 + (15+5i)f_1 + (-1+2i)f_2 + (-19-17i)f_i + (-5+15i)f_{1+i}$$
$$10hf_{1+i}^{(1)} = 5if_0 + (-10+10i)f_1 + (-2-i)f_2 + (2+6i)f_i + (10-20i)f_{1+i}$$

Second Derivative

$$10h^2f_0^{(2)} = (15-55i)f_0 + (60+40i)f_1 + (-9+3i)f_2 + (-36-8i)f_i + (-30+20i)f_{1+i}$$
$$10h^2f_1^{(2)} = (15+5i)f_0 + (-30+10i)f_1 + (9-3i)f_2 + (6-2i)f_i - 10if_{1+i}$$
$$10h^2f_2^{(2)} = (-45+5i)f_0 - 140if_1 + (39+27i)f_2 + (-24+28i)f_i + (30+80i)f_{1+i}$$
$$10h^2f_i^{(2)} = (-45+5i)f_0 + (-30-50i)f_1 + (9-3i)f_2 + (6+58i)f_i + (60-10i)f_{1+i}$$
$$10h^2f_{1+i}^{(2)} = (15+5i)f_0 + 40if_1 + (-9+3i)f_2 + (24-8i)f_i + (-30-40i)f_{1+i}$$

Third Derivative

$$5h^3f_0^{(3)} = (15+45i)f_0 + (-75+15i)f_1 + (6-12i)f_2 + (39-3i)f_i + (15-45i)f_{1+i}$$
$$5h^3f_1^{(3)} = (-15+15i)f_0 + (-15-45i)f_1 + (12+6i)f_2 + (3+9i)f_i + (15+15i)f_{1+i}$$
$$5h^3f_2^{(3)} = (-45-15i)f_0 + (45-105i)f_1 + (18+24i)f_2 + (-33+21i)f_i + (15+75i)f_{1+i}$$
$$5h^3f_i^{(3)} = (45+15i)f_0 + (-15+75i)f_1 + (-12-6i)f_2 + (27-39i)f_i + (-45-45i)f_{1+i}$$
$$5h^3f_{1+i}^{(3)} = (15-15i)f_0 + (45+15i)f_1 + (-6+12i)f_2 + (-9-27i)f_i + (-45+15i)f_{1+i}$$

Fourth Derivative

$$5h^4f_{\text{any}}^{(4)} = (-30-30i)f_0 + (60-60i)f_1 + (6+18i)f_2 + (-36+12i)f_i + 60if_{1+i}$$

Six-Point

First Derivative

$$40hf_0^{(1)} = (-80 + 80i)f_0 + (-32 - 64i)f_1 + (6 - 2i)f_2 + (64 + 32i)f_i + (40 - 40i)f_{1+i} + (2 - 6i)f_{2i}$$

$$40hf_1^{(1)} = (10 - 20i)f_0 + (28 + 76i)f_1 + (1 - 7i)f_2 + (4 - 28i)f_i + (-40 - 20i)f_{1+i} + (-3 - i)f_{2i}$$

$$40hf_2^{(1)} = (-60 - 20i)f_0 + (-32 - 224i)f_1 + (106 + 38i)f_2 + (-96 + 32i)f_i + (80 + 160i)f_{1+i} + (2 + 14i)f_{2i}$$

$$40hf_i^{(1)} = (20 - 10i)f_0 + (28 - 4i)f_1 + (1 + 3i)f_2 + (-76 - 28i)f_i + (20 + 40i)f_{1+i} + (7 - i)f_{2i}$$

$$40hf_{1+i}^{(1)} = (-10 + 10i)f_0 + (-32 + 16i)f_1 + (-4 - 2i)f_2 + (-16 + 32i)f_i + (60 - 60i)f_{1+i} + (2 + 4i)f_{2i}$$

$$40hf_{2i}^{(1)} = (20 + 60i)f_0 + (-32 + 96i)f_1 + (-14 - 2i)f_2 + (224 + 32i)f_i + (-160 - 80i)f_{1+i} + (-38 - 106i)f_{2i}$$

Second Derivative

$$4h^2f_0^{(2)} = -30if_0 + 32f_1 + (-1 + 3i)f_2 - 32f_i + 24if_{1+i} + (1 + 3i)f_{2i}$$

$$4h^2f_1^{(2)} = (7 + 5i)f_0 + (-16 + 8i)f_1 + (3 - 2i)f_2 + 8f_i + (-2 - 10i)f_{1+i} - if_{2i}$$

$$4h^2f_2^{(2)} = (-22 - 20i)f_0 + (32 - 80i)f_1 + (19 + 17i)f_2 - 48f_i + (20 + 76i)f_{1+i} + (-1 + 7i)f_{2i}$$

$$4h^2f_i^{(2)} = (-7 + 5i)f_0 - 8f_1 - if_2 + (16 + 8i)f_i + (2 - 10i)f_{1+i} + (-3 - 2i)f_{2i}$$

$$4h^2f_{1+i}^{(2)} = 4if_0 + (-8 + 8i)f_1 - 2f_2 + (8 + 8i)f_i - 20if_{1+i} + 2f_{2i}$$

$$4h^2f_{2i}^{(2)} = (22 - 20i)f_0 + 48f_1 + (1 + 7i)f_2 + (-32 - 80i)f_i + (-20 + 76i)f_{1+i} + (-19 + 17i)f_{2i}$$

Third Derivative

$$4h^3f_0^{(3)} = (45 + 45i)f_0 + (-48 + 72i)f_1 + (-6 - 9i)f_2 + (72 - 48i)f_i + (-54 - 54i)f_{1+i} + (-9 - 6i)f_{2i}$$

$$4h^3f_1^{(3)} = (-21 + 15i)f_0 + (-24 - 48i)f_1 + (12 + 3i)f_2 + 24if_i + (30 + 6i)f_{1+i} + 3f_{2i}$$

$$4h^3f_2^{(3)} = (-27 - 75i)f_0 + (144 - 120i)f_1 + (18 + 39i)f_2 + (-120 - 48i)f_i + (-6 + 186i)f_{1+i} + (-9 + 18i)f_{2i}$$

$$4h^3f_i^{(3)} = (15 - 21i)f_0 + 24f_1 + 3if_2 + (-48 - 24i)f_i + (6 + 30i)f_{1+i} + (3 + 12i)f_{2i}$$

$$4h^3f_{1+i}^{(3)} = (9 + 9i)f_0 + 24if_1 + (-6 + 3i)f_2 + 24f_i + (-30 - 30i)f_{1+i} + (3 - 6i)f_{2i}$$

$$4h^3f_{2i}^{(3)} = (-75 - 27i)f_0 + (-48 - 120i)f_1 + (18 - 9i)f_2 + (-120 + 144i)f_i + (186 - 6i)f_{1+i} + (39 + 18i)f_{2i}$$

Fourth Derivative

$$h^4f_0^{(4)} = -24f_0 + (-12 - 36i)f_1 + 6f_2 + (-12 + 36i)f_i + 36f_{1+i} + 6f_{2i}$$

$$h^4f_1^{(4)} = (-9 - 15i)f_0 + (24 - 24i)f_1 + (3 + 6i)f_2 - 24f_i + (6 + 30i)f_{1+i} + 3if_{2i}$$

$$h^4 f_2^{(4)} = (6 - 30i)f_0 + (60 - 12i)f_1 + 12if_2 + (-36 - 36i)f_i + (-24 + 60i)f_{1+i} + (-6 + 6i)f_{2i}$$

$$h^4 f_i^{(4)} = (-9 + 15i)f_0 - 24f_1 - 3if_2 + (24 + 24i)f_i + (6 - 30i)f_{1+i} + (3 - 6i)f_{2i}$$

$$h^4 f_{1+i}^{(4)} = 6f_0 + (12 + 12i)f_1 + (-3 + 3i)f_2 + (12 - 12i)f_i - 24f_{1+i} + (-3 - 3i)f_{2i}$$

$$h^4 f_{2i}^{(4)} = (6 + 30i)f_0 + (-36 + 36i)f_1 + (-6 - 6i)f_2 + (60 + 12i)f_i + (-24 - 60i)f_{1+i} - 12if_{2i}$$

Fifth Derivative

$$h^5 f_{\text{any}}^{(5)} = (15 - 15i)f_0 + (36 + 12i)f_1 + (-3 + 6i)f_2 + (-12 - 36i)f_i + (-30 + 30i)f_{1+i} + (-6 + 3i)f_{2i}$$

Seven-Point

First Derivative

$$20hf_0^{(1)} = (-48 + 44i)f_0 + (-40 - 40i)f_1 + (1 - 7i)f_2 + (24 + 32i)f_i + (60 - 20i)f_{1+i} - 8if_{2+i} + (3 - i)f_{2i}$$

$$20hf_1^{(1)} = (5 - 5i)f_0 + (4 + 48i)f_1 + (-3 - 4i)f_2 + (8 - 6i)f_i + (-10 - 30i)f_{1+i} + (-4 - 2i)f_{2+i} - if_{2i}$$

$$20hf_2^{(1)} = (-2 - 14i)f_0 + (48 - 64i)f_1 + (53 + 39i)f_2 + (-8 - 24i)f_i + (-80 + 40i)f_{1+i} + (-8 + 24i)f_{2+i} + (-3 - i)f_{2i}$$

$$20hf_i^{(1)} = (6 - 8i)f_0 + (12 - 16i)f_1 + (3 - i)f_2 + (-48 - 14i)f_i + (20 + 40i)f_{1+i} + (4 - 2i)f_{2+i} + (3 + i)f_{2i}$$

$$20hf_{1+i}^{(1)} = (-1 + 3i)f_0 + (-4 + 12i)f_1 + (-1 + 2i)f_2 + (-4 + 8i)f_i + (10 - 30i)f_{1+i} + 4if_{2+i} + if_{2i}$$

$$20hf_{2+i}^{(1)} = (-8 - 6i)f_0 + (-20 - 40i)f_1 + (-5 - 15i)f_2 + (-20 - 10i)f_i + 100if_{1+i} + (56 - 30i)f_{2+i} + (-3 + i)f_{2i}$$

$$20hf_{2i}^{(1)} = (30 + 10i)f_0 + (64 + 48i)f_1 + (5 + 15i)f_2 + (120 - 40i)f_i - 200f_{1+i} + (8 + 24i)f_{2+i} + (-27 - 57i)f_{2i}$$

Second Derivative

$$20h^2 f_0^{(2)} = (16 - 198i)f_0 + (288 - 64i)f_1 + (27 + 31i)f_2 + (-192 - 96i)f_i + (-160 + 280i)f_{1+i} + (32 + 32i)f_{2+i} + (-11 + 15i)f_{2i}$$

$$20h^2 f_1^{(2)} = (16 + 32i)f_0 + (-132 + 16i)f_1 + (12 - 24i)f_2 + (18 + 34i)f_i + (80 - 40i)f_{1+i} + (2 - 18i)f_{2+i} + 4f_{2i}$$

$$20h^2 f_2^{(2)} = (46 - 68i)f_0 + (408 - 24i)f_1 + (57 + 191i)f_2 + (48 - 136i)f_i + (-460 - 60i)f_{1+i} + (-88 + 112i)f_{2+i} + (-11 - 15i)f_{2i}$$

$$20h^2 f_i^{(2)} = (-24 + 42i)f_0 + (-52 + 56i)f_1 + (-13 + i)f_2 + (118 + 54i)f_i - 140if_{1+i} + (-18 + 2i)f_{2+i} + (-11 - 15i)f_{2i}$$

$$20h^2 f_{1+i}^{(2)} = (6 + 2i)f_0 + (8 + 16i)f_1 + (2 + 6i)f_2 + (28 + 4i)f_i + (-60 - 40i)f_{1+i} + (12 + 12i)f_{2+i} + 4f_{2i}$$

$$20h^2 f_{2+i}^{(2)} = (-54 - 8i)f_0 + (-172 - 144i)f_1 + (-43 - 79i)f_2 + (-122 + 14i)f_i + (300 + 360i)f_{1+i} + (102 - 158i)f_{2+i} + (-11 + 15i)f_{2i}$$

$$20h^2 f_{2i}^{(2)} = (-34 - 168i)f_0 + (88 - 424i)f_1 + (67 - 59i)f_2 + (-472 - 336i)f_i + (340 + 940i)f_{1+i} + (112 - 88i)f_{2+i} + (-101 + 135i)f_{2i}$$

Third Derivative

$$4h^3f_0^{(3)} = (63 + 81i)f_0 + (-84 + 180i)f_1 - 33f_2 + (144 - 12i)f_i + (-54 - 234i)f_{1+i} - 36f_{2+i} - 15if_{2i}$$

$$4h^3f_1^{(3)} = (-27 + 3i)f_0 + (-12 - 84i)f_1 + 21f_2 + (-24 + 12i)f_i + (30 + 66i)f_{1+i} + 12f_{2+i} + 3if_{2i}$$

$$4h^3f_2^{(3)} = (63 - 15i)f_0 + (204 + 180i)f_1 + (-33 + 96i)f_2 + (96 - 60i)f_i + (-246 - 234i)f_{1+i} + (-84 + 48i)f_{2+i} - 15if_{2i}$$

$$4h^3f_i^{(3)} = (9 - 33i)f_0 + (36 - 36i)f_1 + 9f_2 + (-72 - 36i)f_i + (6 + 90i)f_{1+i} + 12f_{2+i} + 15if_{2i}$$

$$4h^3f_{1+i}^{(3)} = (3 - 3i)f_0 + (12 - 12i)f_1 + 3f_2 - 12if_i + (-30 + 30i)f_{1+i} + 12f_{2+i} - 3if_{2i}$$

$$4h^3f_{2+i}^{(3)} = (-39 + 15i)f_0 + (-156 - 36i)f_1 + (-39 - 48i)f_2 + (-72 + 60i)f_i + (294 + 90i)f_{1+i} + (12 - 96i)f_{2+i} + 15if_{2i}$$

$$4h^3f_{2i}^{(3)} = (-81 + 81i)f_0 + (-276 + 84i)f_1 + (-57 - 24i)f_2 + 324if_i + (426 - 426i)f_{1+i} + (-84 - 48i)f_{2+i} + (72 + 9i)f_{2i}$$

Fourth Derivative

$$h^4f_0^{(4)} = (-51 - 9i)f_0 + (-48 - 108i)f_1 + (15 - 18i)f_2 + (-66 + 54i)f_i + (126 + 90i)f_{1+i} + (18 - 18i)f_{2+i} + (6 + 9i)f_{2i}$$

$$h^4f_1^{(4)} = (3 - 21i)f_0 + (60 - 12i)f_1 + (6 + 15i)f_2 + (-12 - 24i)f_i + (-54 + 30i)f_{1+i} + 12if_{2+i} - 3f_{2i}$$

$$h^4f_2^{(4)} = (39 + 21i)f_0 + (24 + 156i)f_1 + (-39 + 30i)f_2 + (78 + 6i)f_i + (-54 - 210i)f_{1+i} + (-54 + 6i)f_{2+i} + (6 - 9i)f_{2i}$$

$$h^4f_i^{(4)} = (-3 + 27i)f_0 + (-36 + 36i)f_1 - 9f_2 + (48 + 36i)f_i + (6 - 90i)f_{1+i} - 12f_{2+i} + (6 - 9i)f_{2i}$$

$$h^4f_{1+i}^{(4)} = (-3 - 3i)f_0 - 12if_1 - 3if_2 + (-6 - 6i)f_i + (6 + 30i)f_{1+i} + (6 - 6i)f_{2+i} - 3f_{2i}$$

$$h^4f_{2+i}^{(4)} = (-21 + 21i)f_0 + (-108 + 12i)f_1 + (-27 - 24i)f_2 + (-24 + 60i)f_i + (186 - 30i)f_{1+i} + (-12 - 48i)f_{2+i} + (6 + 9i)f_{2i}$$

$$h^4f_{2i}^{(4)} = (63 + 9i)f_0 + (120 + 108i)f_1 + (3 + 36i)f_2 + (126 - 90i)f_i + (-294 - 90i)f_{1+i} + (-6 + 54i)f_{2+i} + (-12 - 27i)f_{2i}$$

Fifth Derivative

$$h^5f_0^{(5)} = (63 - 39i)f_0 + (180 + 60i)f_1 + (9 + 42i)f_2 + (36 - 132i)f_i + (-270 + 30i)f_{1+i} + 48if_{2+i} + (-18 - 9i)f_{2i}$$

$$h^5f_1^{(5)} = (45 + 15i)f_0 + (36 + 132i)f_1 + (-27 + 24i)f_2 + (72 - 24i)f_i + (-90 - 150i)f_{1+i} + (-36 + 12i)f_{2+i} - 9if_{2i}$$

$$h^5f_2^{(5)} = (27 + 69i)f_0 + (-108 + 204i)f_1 + (-63 + 6i)f_2 + (108 + 84i)f_i + (90 - 330i)f_{1+i} + (-72 - 24i)f_{2+i} + (18 - 9i)f_{2i}$$

$$h^5f_i^{(5)} = (9 - 57i)f_0 + (108 - 84i)f_1 + (27 + 6i)f_2 + (-72 - 96i)f_i + (-90 + 210i)f_{1+i} + (36 + 12i)f_{2+i} + (-18 + 9i)f_{2i}$$

$$h^5f_{1+i}^{(5)} = (-9 - 3i)f_0 + (-36 - 12i)f_1 + (-9 - 12i)f_2 + (-36 + 12i)f_i + (90 + 30i)f_{1+i} - 24if_{2+i} + 9if_{2i}$$

$$h^5f_{2+i}^{(5)} = (-27 + 51i)f_0 + (-180 + 60i)f_1 + (-45 - 30i)f_2 + 120if_i + (270 - 150i)f_{1+i} + (-36 - 60i)f_{2+i} + (18 + 9i)f_{2i}$$

$$h^5 f_{2i}^{(5)} = (-45 - 75i)f_0 + (36 - 228i)f_1 + (45 - 30i)f_2 + (-180 - 60i)f_i + (90 + 390i)f_{1+i} + (72 - 24i)f_{2+i} + (-18 + 27i)f_{2i}$$

Sixth Derivative

$$h^6 f_{\text{any}}^{(6)} = (-18 + 54i)f_0 + (-144 + 72i)f_1 + (-36 - 18i)f_2 + (36 + 108i)f_i + (180 - 180i)f_{1+i} + (-36 - 36i)f_{2+i} + 18f_{2i}$$

Eight-Point

First Derivative

$$20hf_0^{(1)} = (-52 + 52i)f_0 + (-60 - 20i)f_1 + (-5 - 5i)f_2 + (20 + 60i)f_i + (100 - 100i)f_{1+i} + (-12 - 4i)f_{2+i} + (5 + 5i)f_{2i} + (4 + 12i)f_{1+2i}$$

$$20hf_1^{(1)} = (6 - 2i)f_0 + (4 + 58i)f_1 + (-4 - 2i)f_2 + (14 + 2i)f_i + (-20 - 60i)f_{1+i} + (-6 + 2i)f_{2+i} + 2f_{2i} + (4 + 2i)f_{1+2i}$$

$$20hf_2^{(1)} = (10 - 10i)f_0 + (80 - 40i)f_1 + (57 + 47i)f_2 + (32 - 24i)f_i - 200f_{1+i} + 40if_{2+i} + (5 - 5i)f_{2i} + (16 - 8i)f_{1+2i}$$

$$20hf_i^{(1)} = (2 - 6i)f_0 + (-2 - 14i)f_1 - 2if_2 + (-58 - 4i)f_i + (60 + 20i)f_{1+i} + (-2 - 4i)f_{2+i} + (2 + 4i)f_{2i} + (-2 + 6i)f_{1+2i}$$

$$20hf_{1+i}^{(1)} = (-1 + i)f_0 + (-2 + 6i)f_1 + if_2 + (-6 + 2i)f_i + (10 - 10i)f_{1+i} + (2 + 2i)f_{2+i} - f_{2i} + (-2 - 2i)f_{1+2i}$$

$$20hf_{2+i}^{(1)} = (2 - 6i)f_0 + (10 - 30i)f_1 - 10if_2 + (10 - 20i)f_i + (-100 + 100i)f_{1+i} + (66 - 20i)f_{2+i} + (2 - 4i)f_{2i} + (10 - 10i)f_{1+2i}$$

$$20hf_{2i}^{(1)} = (10 - 10i)f_0 + (24 - 32i)f_1 + (5 - 5i)f_2 + (40 - 80i)f_i + 200if_{1+i} + (8 - 16i)f_{2+i} + (-47 - 57i)f_{2i} - 40f_{1+2i}$$

$$20hf_{1+2i}^{(1)} = (6 - 2i)f_0 + (20 - 10i)f_1 + (4 - 2i)f_2 + (30 - 10i)f_i + (-100 + 100i)f_{1+i} + (10 - 10i)f_{2+i} + 10f_{2i} + (20 - 66i)f_{1+2i}$$

Second Derivative

$$20h^2 f_0^{(2)} = -254if_0 + (296 - 248i)f_1 + (47 - 5i)f_2 + (-296 - 248i)f_i + 840if_{1+i} + (72 - 40i)f_{2+i} + (-47 - 5i)f_{2i} + (-72 - 40i)f_{1+2i}$$

$$20h^2 f_1^{(2)} = (2 + 38i)f_0 + (-180 + 20i)f_1 + (2 - 28i)f_2 + (-18 + 66i)f_i + (220 - 100i)f_{1+i} + (-18 - 26i)f_{2+i} + 10if_{2i} + (-8 + 20i)f_{1+2i}$$

$$20h^2 f_2^{(2)} = 94f_0 + (484 + 228i)f_1 + (47 + 249i)f_2 + (260 + 20i)f_i + (-940 - 740i)f_{1+i} + (-108 + 228i)f_{2+i} + (47 - 5i)f_{2i} + (116 + 20i)f_{1+2i}$$

$$20h^2 f_i^{(2)} = (-2 + 38i)f_0 + (18 + 66i)f_1 + 10if_2 + (180 + 20i)f_i + (-220 - 100i)f_{1+i} + (8 + 20i)f_{2+i} + (-2 - 28i)f_{2i} + (18 - 26i)f_{1+2i}$$

$$20h^2 f_{1+i}^{(2)} = (-8 + 4i)f_1 + 2if_2 + (8 + 4i)f_i - 20if_{1+i} + (8 + 4i)f_{2+i} + 2if_{2i} + (-8 + 4i)f_{1+2i}$$

$$20h^2 f_{2+i}^{(2)} = (2 - 38i)f_0 + (26 - 178i)f_1 - 66if_2 + (16 - 132i)f_i + (-260 + 660i)f_{1+i} + (188 - 132i)f_{2+i} + (2 - 28i)f_{2i} + (26 - 86i)f_{1+2i}$$

$$20h^2 f_{2i}^{(2)} = -94f_0 + (-260 + 20i)f_1 + (-47 - 5i)f_2 + (-484 + 228i)f_i + (940 - 740i)f_{1+i} + (-116 + 20i)f_{2+i} + (-47 + 249i)f_{2i} + (108 + 228i)f_{1+2i}$$

$$20h^3f^{(2)}_{1+2i} = (-2-38i)f_0 + (-16-132i)f_1 + (-2-28i)f_2 + (-26-178i)f_i + (260+660i)f_{1+i} + (-26-86i)f_{2+i} - 66if_{2i} + (-188-132i)f_{1+2i}$$

Third Derivative

$$20h^3f^{(3)}_0 = (543+543i)f_0 + (126+1542i)f_1 + (-120+183i)f_2 + (1542+126i)f_i + (-2550-2550i)f_{1+i} + (-90+366i)f_{2+i} + (183-120i)f_{2i} + (366-90i)f_{1+2i}$$

$$20h^3f^{(3)}_1 = (-162-42i)f_0 + (-84-618i)f_1 + (120-42i)f_2 + (-258-84i)f_i + (420+900i)f_{1+i} + (90-84i)f_{2+i} - 42f_{2i} + (-84-30i)f_{1+2i}$$

$$20h^3f^{(3)}_2 = (303+303i)f_0 + (606+2022i)f_1 + (-360+663i)f_2 + (822+846i)f_i + (-1110-4950i)f_{1+i} + (-810+606i)f_{2+i} + (183+120i)f_{2i} + (366+390i)f_{1+2i}$$

$$20h^3f^{(3)}_i = (-42-162i)f_0 + (-84-258i)f_1 - 42if_2 + (-618-84i)f_i + (900+420i)f_{1+i} + (-30-84i)f_{2+i} + (-42+120i)f_{2i} + (-84+90i)f_{1+2i}$$

$$20h^3f^{(3)}_{1+i} = (3+3i)f_0 + (6-18i)f_1 + 3if_2 + (-18+6i)f_i + (-30-30i)f_{1+i} + (30+6i)f_{2+i} + 3f_{2i} + (6+30i)f_{1+2i}$$

$$20h^3f^{(3)}_{2+i} = (-42-162i)f_0 + (-84-738i)f_1 - 282if_2 + (-138-564i)f_i + (-60+2820i)f_{1+i} + (450-564i)f_{2+i} + (-42-120i)f_{2i} + (-84-390i)f_{1+2i}$$

$$20h^3f^{(3)}_{2i} = (303+303i)f_0 + (846+822i)f_1 + (120+183i)f_2 + (2022+606i)f_i + (-4950-1110i)f_{1+i} + (390+366i)f_{2+i} + (663-360i)f_{2i} + (606-810i)f_{1+2i}$$

$$20h^3f^{(3)}_{1+2i} = (-162-42i)f_0 + (-564-138i)f_1 + (-120-42i)f_2 + (-738-84i)f_i + (2820-60i)f_{1+i} + (-390-84i)f_{2+i} - 282f_{2i} + (-564+450i)f_{1+2i}$$

Fourth Derivative

$$h^4f^{(4)}_0 = -96f_0 + (-192-126i)f_1 + (-12-36i)f_2 + (-192+126i)f_i + 576f_{1+i} + (-36-54i)f_{2+i} + (-12+36i)f_{2i} + (-36+54i)f_{1+2i}$$

$$h^4f^{(4)}_1 = (15-27i)f_0 + (102-18i)f_1 + (15+18i)f_2 + (18-54i)f_i + (-174+90i)f_{1+i} + (18+18i)f_{2+i} - 9if_{2i} + (6-18i)f_{1+2i}$$

$$h^4f^{(4)}_2 = 72if_0 + (-144+270i)f_1 + (-84+36i)f_2 + (12+198i)f_i + (336-720i)f_{1+i} + (-144+18i)f_{2+i} + (12+36i)f_{2i} + (12+90i)f_{1+2i}$$

$$h^4f^{(4)}_i = (15+27i)f_0 + (18+54i)f_1 + 9if_2 + (102+18i)f_i + (-174-90i)f_{1+i} + (6+18i)f_{2+i} + (15-18i)f_{2i} + (18-18i)f_{1+2i}$$

$$h^4f^{(4)}_{1+i} = 6f_1 + 6f_i - 24f_{1+i} + 6f_{2+i} + 6f_{1+2i}$$

$$h^4f^{(4)}_{2+i} = (-15-27i)f_0 + (-42-126i)f_1 - 45if_2 + (-54-90i)f_i + (126+450i)f_{1+i} + (42-90i)f_{2+i} + (-15-18i)f_{2i} + (-42-54i)f_{1+2i}$$

$$h^4f^{(4)}_{2i} = -72if_0 + (12-198i)f_1 + (12-36i)f_2 + (-144-270i)f_i + (336+720i)f_{1+i} + (12-90i)f_{2+i} + (-84-36i)f_{2i} + (-144-18i)f_{1+2i}$$

$$h^4 f^{(4)}_{1+2i} = (-15+27i)f_0 + (-54+90i)f_1 + (-15+18i)f_2 + (-42+126i)f_i + (126-450i)f_{1+i} + (-42+54i)f_{2+i} + 45if_{2i} + (42+90i)f_{1+2i}$$

Fifth Derivative

$$h^5 f^{(5)}_0 = (132-132i)f_0 + (480-150i)f_1 + (90+30i)f_2 + (150-480i)f_i + (-960+960i)f_{1+i} + (162+24i)f_{2+i} + (-30-90i)f_{2i} + (-24-162i)f_{1+2i}$$

$$h^5 f^{(5)}_1 = (69+57i)f_0 + (66+282i)f_1 + (-36+57i)f_2 + (186+78i)f_i + (-330-570i)f_{1+i} + (-54+78i)f_{2+i} + 33f_{2i} + (66+18i)f_{1+2i}$$

$$h^5 f^{(5)}_2 = (-120+120i)f_0 + (-600+210i)f_1 + (-162-42i)f_2 + (-282+384i)f_i + (1560-840i)f_{1+i} + (-270-120i)f_{2+i} + (-30+90i)f_{2i} + (-96+198i)f_{1+2i}$$

$$h^5 f^{(5)}_i = (-57-69i)f_0 + (-78-186i)f_1 - 33if_2 + (-282-66i)f_i + (570+330i)f_{1+i} + (-18-66i)f_{2+i} + (-57+36i)f_{2i} + (-78+54i)f_{1+2i}$$

$$h^5 f^{(5)}_{1+i} = (6-6i)f_0 + (12-6i)f_1 - 6if_2 + (6-12i)f_i + (-60+60i)f_{1+i} + (18-12i)f_{2+i} + 6f_{2i} + (12-18i)f_{1+2i}$$

$$h^5 f^{(5)}_{2+i} = (-57-69i)f_0 + (-150-330i)f_1 - 105if_2 + (-210-210i)f_i + (570+1050i)f_{1+i} + (54-210i)f_{2+i} + (-57-36i)f_{2i} + (-150-90i)f_{1+2i}$$

$$h^5 f^{(5)}_{2i} = (-120+120i)f_0 + (-384+282i)f_1 + (-90+30i)f_2 + (-210+600i)f_i + (840-1560i)f_{1+i} + (-198+96i)f_{2+i} + (42+162i)f_{2i} + (120+270i)f_{1+2i}$$

$$h^5 f^{(5)}_{1+2i} = (69+57i)f_0 + (210+210i)f_1 + (36+57i)f_2 + (330+150i)f_i + (-1050-570i)f_{1+i} + (90+150i)f_{2+i} + 105f_{2i} + (210-54i)f_{1+2i}$$

Sixth Derivative

$$h^6 f^{(6)}_0 = 252if_0 + (-288+684i)f_1 + (-126+90i)f_2 + (288+684i)f_i - 2160if_{1+i} + (-216+180i)f_{2+i} + (126+90i)f_{2i} + (216+180i)f_{1+2i}$$

$$h^6 f^{(6)}_1 = (-126+126i)f_0 + (-540+180i)f_1 + (-126-36i)f_2 + (-216+432i)f_i + (1260-900i)f_{1+i} + (-216-72i)f_{2+i} + 90if_{2i} + (-36+180i)f_{1+2i}$$

$$h^6 f^{(6)}_2 = -252f_0 + (-792-324i)f_1 + (-126-162i)f_2 + (-720+180i)f_i + (2520+360i)f_{1+i} + (-216-324i)f_{2+i} + (-126+90i)f_{2i} + (-288+180i)f_{1+2i}$$

$$h^6 f^{(6)}_i = (126+126i)f_0 + (216+432i)f_1 + 90if_2 + (540+180i)f_i + (-1260-900i)f_{1+i} + (36+180i)f_{2+i} + (126-36i)f_{2i} + (216-72i)f_{1+2i}$$

$$h^6 f^{(6)}_{1+i} = (-36-72i)f_1 - 36if_2 + (36-72i)f_i + 360if_{1+i} + (36-72i)f_{2+i} - 36if_{2i} + (-36-72i)f_{1+2i}$$

$$h^6 f^{(6)}_{2+i} = (-126 - 126i)f_0 + (-288 - 576i)f_1 - 162if_2 + (-468 - 324i)f_i + (1260 + 1620i)f_{1+i} + (36 - 324i)f_{2+i} + (-126 - 36i)f_{2i} + (-288 - 72i)f_{1+2i}$$

$$h^6 f^{(6)}_{2i} = 252f_0 + (720 + 180i)f_1 + (126 + 90i)f_2 + (792 - 324i)f_i + (-2520 + 360i)f_{1+i} + (288 + 180i)f_{2+i} + (126 - 162i)f_{2i} + (216 - 324i)f_{1+2i}$$

$$h^6 f^{(6)}_{1+2i} = (126 - 126i)f_0 + (468 - 324i)f_1 + (126 - 36i)f_2 + (288 - 576i)f_i + (-1260 + 1620i)f_{1+i} + (288 - 72i)f_{2+i} - 162if_{2i} + (-36 - 324i)f_{1+2i}$$

Seventh Derivative

$$h^7 f^{(7)}_{\text{any}} = (-126 - 126i)f_0 + (-252 - 504i)f_1 - 126if_2 + (-504 - 252i)f_i + (1260 + 1260i)f_{1+i} - 252if_{2+i} - 126f_{2i} - 252f_{1+2i}$$

Nine-Point

First Derivative

$$40hf^{(1)}_0 = (-114 + 114i)f_0 - 160f_1 - 20f_2 + 160if_i + (400 - 400i)f_{1+i} + (-64 + 32i)f_{2+i} + 20if_{2i} + (-32 + 64i)f_{1+2i} + (-10 + 10i)f_{2+2i}$$

$$40hf^{(1)}_1 = 10f_0 + 132if_1 - 10f_2 + (20 + 20i)f_i - 200if_{1+i} + (-20 + 20i)f_{2+i} + (2 + 4i)f_{2i} + 20if_{1+2i} + (-2 + 4i)f_{2+2i}$$

$$40hf^{(1)}_2 = 20f_0 + 160f_1 + (114 + 114i)f_2 + (64 + 32i)f_i + (-400 - 400i)f_{1+i} + 160if_{2+i} + (10 + 10i)f_{2i} + (32 + 64i)f_{1+2i} + 20if_{2+2i}$$

$$40hf^{(1)}_i = -10if_0 + (-20 - 20i)f_1 + (-4 - 2i)f_2 - 132f_i + 200f_{1+i} - 20f_{2+i} + 10if_{2i} + (-20 + 20i)f_{1+2i} + (-4 + 2i)f_{2+2i}$$

$$40hf^{(1)}_{1+i} = (-1 + i)f_0 + 8if_1 + (1 + i)f_2 - 8f_i + 8f_{2+i} + (-1 - i)f_{2i} - 8if_{1+2i} + (1 - i)f_{2+2i}$$

$$40hf^{(1)}_{2+i} = (4 - 2i)f_0 + (20 - 20i)f_1 - 10if_2 + 20f_i - 200f_{1+i} + 132f_{2+i} + (4 + 2i)f_{2i} + (20 + 20i)f_{1+2i} + 10if_{2+2i}$$

$$40hf^{(1)}_{2i} = -20if_0 + (-32 - 64i)f_1 + (-10 - 10i)f_2 - 160if_i + (400 + 400i)f_{1+i} + (-64 - 32i)f_{2+i} + (-114 - 114i)f_{2i} - 160f_{1+2i} - 20f_{2+2i}$$

$$40hf^{(1)}_{1+2i} = (2 - 4i)f_0 - 20if_1 + (-2 - 4i)f_2 + (20 - 20i)f_i + 200if_{1+i} + (-20 - 20i)f_{2+i} + 10f_{2i} - 132if_{1+2i} - 10f_{2+2i}$$

$$40hf^{(1)}_{2+2i} = (10 - 10i)f_0 + (32 - 64i)f_1 - 20if_2 + (64 - 32i)f_i + (-400 + 400i)f_{1+i} - 160if_{2+i} + 20f_{2i} + 160f_{1+2i} + (114 - 114i)f_{2+2i}$$

Second Derivative

$$20h^2 f^{(2)}_0 = -306if_0 + (296 - 456i)f_1 + (47 - 57i)f_2 + (-296 - 456i)f_i + 1880if_{1+i} + (72 - 248i)f_{2+i} + (-47 - 57i)f_{2i} + (-72 - 248i)f_{1+2i} - 52if_{2+2i}$$

$$20h^2 f^{(2)}_1 = (-10 + 33i)f_0 - 228f_1 + (-10 - 33i)f_2 + (-66 + 46i)f_i + 460f_{1+i} + (-66 - 46i)f_{2+i} + (-12 + 5i)f_{2i} - 56f_{1+2i} + (-12 - 5i)f_{2+2i}$$

$$20h^2 f^{(2)}_2 = (47 + 57i)f_0 + (296 + 456i)f_1 + 306if_2 + (72 + 248i)f_i - 1880if_{1+i} + (-296 + 456i)f_{2+i} + 52if_{2i} + (-72 + 248i)f_{1+2i} + (-47 + 57i)f_{2+2i}$$

$$20h^2f_i^{(2)} = (10+33i)f_0+(66+46i)f_1+(12+5i)f_2+228f_i-460f_{1+i}+56f_{2+i}+(10-33i)f_{2i}+(66-46i)f_{1+2i}+(12-5i)f_{2+2i}$$

$$20h^2f_{1+i}^{(2)} = -if_0-8f_1+if_2+8f_i+8f_{2+i}+if_{2i}-8f_{1+2i}-if_{2+2i}$$

$$20h^2f_{2+i}^{(2)} = (12-5i)f_0+(66-46i)f_1+(10-33i)f_2+56f_i-460f_{1+i}+228f_{2+i}+(12+5i)f_{2i}+(66+46i)f_{1+2i}+(10+33i)f_{2+2i}$$

$$20h^2f_{2i}^{(2)} = (-47+57i)f_0+(-72+248i)f_1+52if_2+(-296+456i)f_i-1880if_{1+i}+(72+248i)f_{2+i}+306if_{2i}+(296+456i)f_{1+2i}+(47+57i)f_{2+2i}$$

$$20h^2f_{1+2i}^{(2)} = (-12-5i)f_0-56f_1+(-12+5i)f_2+(-66-46i)f_i+460f_{1+i}+(-66+46i)f_{2+i}+(-10-33i)f_{2i}-228f_{1+2i}+(-10+33i)f_{2+2i}$$

$$20h^2f_{2+2i}^{(2)} = -52if_0+(-72-248i)f_1+(-47-57i)f_2+(72-248i)f_i+1880if_{1+i}+(-296-456i)f_{2+i}+(47-57i)f_{2i}+(296-456i)f_{1+2i}-306if_{2+2i}$$

Third Derivative

$$40h^3f_0^{(3)} = (1467+1467i)f_0+(1776+4608i)f_1+(141+747i)f_2+(4608+1776i)f_i+(-12720-12720i)f_{1+i}+(1344+2256i)f_{2+i}+(747+141i)f_{2i}+(2256+1344i)f_{1+2i}+(381+381i)f_{2+2i}$$

$$40h^3f_1^{(3)} = (-282-198i)f_0-1692if_1+(282-198i)f_2+(-348-624i)f_i+4080if_{1+i}+(348-624i)f_{2+i}+(-42-114i)f_{2i}-516if_{1+2i}+(42-114i)f_{2+2i}$$

$$40h^3f_2^{(3)} = (-141+747i)f_0+(-1776+4608i)f_1+(-1467+1467i)f_2+(-1344+2256i)f_i+(12720-12720i)f_{1+i}+(-4608+1776i)f_{2+i}+(-381+381i)f_{2i}+(-2256+1344i)f_{1+2i}+(-747+141i)f_{2+2i}$$

$$40h^3f_i^{(3)} = (-198-282i)f_0+(-624-348i)f_1+(-114-42i)f_2-1692f_i+4080f_{1+i}-516f_{2+i}+(-198+282i)f_{2i}+(-624+348i)f_{1+2i}+(-114+42i)f_{2+2i}$$

$$40h^3f_{1+i}^{(3)} = (3+3i)f_0-48if_1+(-3+3i)f_2-48f_i+48f_{2+i}+(3-3i)f_{2i}+48if_{1+2i}+(-3-3i)f_{2+2i}$$

$$40h^3f_{2+i}^{(3)} = (114-42i)f_0+(624-348i)f_1+(198-282i)f_2+516f_i-4080f_{1+i}+1692f_{2+i}+(114+42i)f_{2i}+(624+348i)f_{1+2i}+(198+282i)f_{2+2i}$$

$$40h^3f_{2i}^{(3)} = (747-141i)f_0+(2256-1344i)f_1+(381-381i)f_2+(4608-1776i)f_i+(-12720+12720i)f_{1+i}+(1344-2256i)f_{2+i}+(1467-1467i)f_{2i}+(1776-4608i)f_{1+2i}+(141-747i)f_{2+2i}$$

$$40h^3f_{1+2i}^{(3)} = (-42+114i)f_0+516if_1+(42+114i)f_2+(-348+624i)f_i-4080if_{1+i}+(348+624i)f_{2+i}+(-282+198i)f_{2i}+1692if_{1+2i}+(282+198i)f_{2+2i}$$

$$40h^3f_{2+2i}^{(3)} = (-381-381i)f_0+(-2256-1344i)f_1+(-747-141i)f_2+(-1344-2256i)f_i+(12720+12720i)f_{1+i}+(-4608-1776i)f_{2+i}+(-141-747i)f_{2i}+(-1776-4608i)f_{1+2i}+(-1467-1467i)f_{2+2i}$$

Fourth Derivative

$$10h^4f_0^{(4)} = -1503f_0+(-4092-1260i)f_1+(-663-360i)f_2+(-4092+1260i)f_i+16620f_{1+i}+(-2532-540i)f_{2+i}+(-663+360i)f_{2i}+(-2532+540i)f_{1+2i}-543f_{2+2i}$$

$$10h^4f_1^{(4)} = (282 - 225i)f_0 + 1548f_1 + (282 + 225i)f_2 + (708 - 360i)f_i - 4380f_{1+i} + (708 + 360i)f_{2+i} + (132 - 45i)f_{2i} + 588f_{1+2i} + (132 + 45i)f_{2+2i}$$

$$10h^4f_2^{(4)} = (-663 + 360i)f_0 + (-4092 + 1260i)f_1 - 1503f_2 + (-2532 + 540i)f_i + 16620f_{1+i} + (-4092 - 1260i)f_{2+i} - 543f_{2i} + (-2532 - 540i)f_{1+2i} + (-663 - 360i)f_{2+2i}$$

$$10h^4f_i^{(4)} = (282 + 225i)f_0 + (708 + 360i)f_1 + (132 + 45i)f_2 + 1548f_i - 4380f_{1+i} + 588f_{2+i} + (282 - 225i)f_{2i} + (708 - 360i)f_{1+2i} + (132 - 45i)f_{2+2i}$$

$$10h^4f_{1+i}^{(4)} = -3f_0 + 48f_1 - 3f_2 + 48f_i - 180f_{1+i} + 48f_{2+i} - 3f_{2i} + 48f_{1+2i} - 3f_{2+2i}$$

$$10h^4f_{2+i}^{(4)} = (132 - 45i)f_0 + (708 - 360i)f_1 + (282 - 225i)f_2 + 588f_i - 4380f_{1+i} + 1548f_{2+i} + (132 + 45i)f_{2i} + (708 + 360i)f_{1+2i} + (282 + 225i)f_{2+2i}$$

$$10h^4f_{2i}^{(4)} = (-663 - 360i)f_0 + (-2532 - 540i)f_1 - 543f_2 + (-4092 - 1260i)f_i + 16620f_{1+i} + (-2532 + 540i)f_{2+i} - 1503f_{2i} + (-4092 + 1260i)f_{1+2i} + (-663 + 360i)f_{2+2i}$$

$$10h^4f_{1+2i}^{(4)} = (132 + 45i)f_0 + 588f_1 + (132 - 45i)f_2 + (708 + 360i)f_i - 4380f_{1+i} + (708 - 360i)f_{2+i} + (282 + 225i)f_{2i} + 1548f_{1+2i} + (282 - 225i)f_{2+2i}$$

$$10h^4f_{2+2i}^{(4)} = -543f_0 + (-2532 + 540i)f_1 + (-663 + 360i)f_2 + (-2532 - 540i)f_i + 16620f_{1+i} + (-4092 + 1260i)f_{2+i} + (-663 - 360i)f_{2i} + (-4092 - 1260i)f_{1+2i} - 1503f_{2+2i}$$

Fifth Derivative

$$2h^5f_0^{(5)} = (504 - 504i)f_0 + (1920 - 1260i)f_1 + (420 - 180i)f_2 + (1260 - 1920i)f_i + (-6720 + 6720i)f_{1+i} + (1284 - 912i)f_{2+i} + (180 - 420i)f_{2i} + (912 - 1284i)f_{1+2i} + (240 - 240i)f_{2+2i}$$

$$2h^5f_1^{(5)} = (105 + 225i)f_0 + 1008if_1 + (-105 + 225i)f_2 + (240 + 600i)f_i - 3360if_{1+i} + (-240 + 600i)f_{2+i} + (33 + 111i)f_{2i} + 480if_{1+2i} + (-33 + 111i)f_{2+2i}$$

$$2h^5f_2^{(5)} = (-420 - 180i)f_0 + (-1920 - 1260i)f_1 + (-504 - 504i)f_2 + (-1284 - 912i)f_i + (6720 + 6720i)f_{1+i} + (-1260 - 1920i)f_{2+i} + (-240 - 240i)f_{2i} + (-912 - 1284i)f_{1+2i} + (-180 - 420i)f_{2+2i}$$

$$2h^5f_i^{(5)} = (-225 - 105i)f_0 + (-600 - 240i)f_1 + (-111 - 33i)f_2 - 1008f_i + 3360f_{1+i} - 480f_{2+i} + (-225 + 105i)f_{2i} + (-600 + 240i)f_{1+2i} + (-111 + 33i)f_{2+2i}$$

$$2h^5f_{1+i}^{(5)} = (6 - 6i)f_0 + 12if_1 + (-6 - 6i)f_2 - 12f_i + 12f_{2+i} + (6 + 6i)f_{2i} - 12if_{1+2i} + (-6 + 6i)f_{2+2i}$$

$$2h^5f_{2+i}^{(5)} = (111 - 33i)f_0 + (600 - 240i)f_1 + (225 - 105i)f_2 + 480f_i - 3360f_{1+i} + 1008f_{2+i} + (111 + 33i)f_{2i} + (600 + 240i)f_{1+2i} + (225 + 105i)f_{2+2i}$$

$$2h^5f_{2i}^{(5)} = (180 + 420i)f_0 + (912 + 1284i)f_1 + (240 + 240i)f_2 + (1260 + 1920i)f_i + (-6720 - 6720i)f_{1+i} + (1284 + 912i)f_{2+i} + (504 + 504i)f_{2i} + (1920 + 1260i)f_{1+2i} + (420 + 180i)f_{2+2i}$$

$$2h^5f^{(5)}_{1+2i} = (33 - 111i)f_0 - 480if_1 + (-33 - 111i)f_2 + (240 - 600i)f_i + 3360if_{1+i} + (-240 - 600i)f_{2+i} + (105 - 225i)f_{2i} - 1008if_{1+2i} + (-105 - 225i)f_{2+2i}$$

$$2h^5f^{(5)}_{2+2i} = (-240 + 240i)f_0 + (-912 + 1284i)f_1 + (-180 + 420i)f_2 + (-1284 + 912i)f_i + (6720 - 6720i)f_{1+i} + (-1260 + 1920i)f_{2+i} + (-420 + 180i)f_{2i} + (-1920 + 1260i)f_{1+2i} + (-504 + 504i)f_{2+2i}$$

Sixth Derivative

$$h^6f^{(6)}_0 = 648if_0 + (-288 + 2268i)f_1 + (-126 + 486i)f_2 + (288 + 2268i)f_i - 10080if_{1+i} + (-216 + 1764i)f_{2+i} + (126 + 486i)f_{2i} + (216 + 1764i)f_{1+2i} + 396if_{2+2i}$$

$$h^6f^{(6)}_1 = (-315 + 81i)f_0 - 1296f_1 + (-315 - 81i)f_2 + (-972 + 252i)f_i + 5040f_{1+i} + (-972 - 252i)f_{2+i} + (-189 + 45i)f_{2i} - 792f_{1+2i} + (-189 - 45i)f_{2+2i}$$

$$h^6f^{(6)}_2 = (-126 - 486i)f_0 + (-288 - 2268i)f_1 - 648if_2 + (-216 - 1764i)f_i + 10080if_{1+i} + (288 - 2268i)f_{2+i} - 396if_{2i} + (216 - 1764i)f_{1+2i} + (126 - 486i)f_{2+2i}$$

$$h^6f^{(6)}_i = (315 + 81i)f_0 + (972 + 252i)f_1 + (189 + 45i)f_2 + 1296f_i - 5040f_{1+i} + 792f_{2+i} + (315 - 81i)f_{2i} + (972 - 252i)f_{1+2i} + (189 - 45i)f_{2+2i}$$

$$h^6f^{(6)}_{1+i} = 18if_0 - 36f_1 - 18if_2 + 36f_i + 36f_{2+i} - 18if_{2i} - 36f_{1+2i} + 18if_{2+2i}$$

$$h^6f^{(6)}_{2+i} = (189 - 45i)f_0 + (972 - 252i)f_1 + (315 - 81i)f_2 + 792f_i - 5040f_{1+i} + 1296f_{2+i} + (189 + 45i)f_{2i} + (972 + 252i)f_{1+2i} + (315 + 81i)f_{2+2i}$$

$$h^6f^{(6)}_{2i} = (126 - 486i)f_0 + (216 - 1764i)f_1 - 396if_2 + (288 - 2268i)f_i + 10080if_{1+i} + (-216 - 1764i)f_{2+i} - 648if_{2i} + (-288 - 2268i)f_{1+2i} + (-126 - 486i)f_{2+2i}$$

$$h^6f^{(6)}_{1+2i} = (-189 - 45i)f_0 - 792f_1 + (-189 + 45i)f_2 + (-972 - 252i)f_i + 5040f_{1+i} + (-972 + 252i)f_{2+i} + (-315 - 81i)f_{2i} - 1296f_{1+2i} + (-315 + 81i)f_{2+2i}$$

$$h^6f^{(6)}_{2+2i} = 396if_0 + (216 + 1764i)f_1 + (126 + 486i)f_2 + (-216 + 1764i)f_i - 10080if_{1+i} + (288 + 2268i)f_{2+i} + (-126 + 486i)f_{2i} + (-288 + 2268i)f_{1+2i} + 648if_{2+2i}$$

Seventh Derivative

$$h^7f^{(7)}_0 = (-567 - 567i)f_0 + (-2016 - 2268i)f_1 + (-441 - 567i)f_2 + (-2268 - 2016i)f_i + (10080 + 10080i)f_{1+i} + (-1764 - 2016i)f_{2+i} + (-567 - 441i)f_{2i} + (-2016 - 1764i)f_{1+2i} + (-441 - 441i)f_{2+2i}$$

$$h^7f^{(7)}_1 = (-63 - 567i)f_0 - 2268if_1 + (63 - 567i)f_2 + (-252 - 2016i)f_i + 10080if_{1+i} + (252 - 2016i)f_{2+i} + (-63 - 441i)f_{2i} - 1764if_{1+2i} + (63 - 441i)f_{2+2i}$$

$$h^7f^{(7)}_2 = (441 - 567i)f_0 + (2016 - 2268i)f_1 + (567 - 567i)f_2 + (1764 - 2016i)f_i + (-10080 + 10080i)f_{1+i} + (2268 - 2016i)f_{2+i} + (441 - 441i)f_{2i} + (2016 - 1764i)f_{1+2i} + (567 - 441i)f_{2+2i}$$

$$h^7f^{(7)}_i = (-567 - 63i)f_0 + (-2016 - 252i)f_1 + (-441 - 63i)f_2 - 2268f_i + 10080f_{1+i} - 1764f_{2+i} + (-567 + 63i)f_{2i} + (-2016 + 252i)f_{1+2i} + (-441 + 63i)f_{2+2i}$$

$$h^7 f^{(7)}_{1+i} = (-63 - 63i)f_0 - 252if_1 + (63 - 63i)f_2 - 252f_i + 252f_{2+i} + (-63 + 63i)f_{2i} + 252if_{1+2i} + (63 + 63i)f_{2+2i}$$

$$h^7 f^{(7)}_{2+i} = (441 - 63i)f_0 + (2016 - 252i)f_1 + (567 - 63i)f_2 + 1764f_i - 10080f_{1+i} + 2268f_{2+i} + (441 + 63i)f_{2i} + (2016 + 252i)f_{1+2i} + (567 + 63i)f_{2+2i}$$

$$h^7 f^{(7)}_{2i} = (-567 + 441i)f_0 + (-2016 + 1764i)f_1 + (-441 + 441i)f_2 + (-2268 + 2016i)f_i + (10080 - 10080i)f_{1+i} + (-1764 + 2016i)f_{2+i} + (-567 + 567i)f_{2i} + (-2016 + 2268i)f_{1+2i} + (-441 + 567i)f_{2+2i}$$

$$h^7 f^{(7)}_{1+2i} = (-63 + 441i)f_0 + 1764if_1 + (63 + 441i)f_2 + (-252 + 2016i)f_i - 10080if_{1+i} + (252 + 2016i)f_{2+i} + (-63 + 567i)f_{2i} + 2268if_{1+2i} + (63 + 567i)f_{2+2i}$$

$$h^7 f^{(7)}_{2+2i} = (441 + 441i)f_0 + (2016 + 1764i)f_1 + (567 + 441i)f_2 + (1764 + 2016i)f_i + (-10080 - 10080i)f_{1+i} + (2268 + 2016i)f_{2+i} + (441 + 567i)f_{2i} + (2016 + 2268i)f_{1+2i} + (567 + 567i)f_{2+2i}$$

Eighth Derivative

$$h^8 f^{(8)}_{\text{any}} = 504f_0 + 2016f_1 + 504f_2 + 2016f_i - 10080f_{1+i} + 2016f_{2+i} + 504f_{2i} + 2016f_{1+2i} + 504f_{2+2i}$$

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THE ZEROS OF $P'_{n_i}(x_0)$ OF NON-INTEGRAL DEGREE

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1. Introduction. In January, 1951, tables were published [1] on the zeros of $P'_{n_i}(x)$, the Associated Legendre Function of order 1 and non-integral degree. In April, 1951, certain errors were observed [2] in those tables of Reference 1.

This note explains a method devised for use at the Willow Run Research Center which, though not as accurate as the method described in Reference 2, can still yield results suitable for engineering problems. Throughout we will consider the general problem for n_i real and $x_0 = \cos \theta_0$. We will continually use as a numerical example $x_0 = \cos 165°$.

We are interested in finding the values of n_i and $\int_{x_0}^{1} [P'_{n_i}(x)]^2 \, dx$ where the n_i's are such that

$$P'_{n_i}(x_0) = 0 \tag{1.1}$$

In order to introduce as many bounds and checks of results as possible, the following asymptotic relations and observations were employed.

When the n_i are large, we obtain from the asymptotic formula [3]:

$$n_{i+1} - n_i = \pi/\theta_0$$

The two limiting cases for flow around a cone or scattering by a cone are $\theta_0 = \pi$ and $\theta_0 = \pi/2$, and for these two limiting cases the n_i differ by 1 and 2, respectively.

Thus for $\theta_0 = 165°$ the maximum value of the differences between successive n_i's is 1.0909. A few asymptotic difference values are shown in Table 1.

In the 90° and 180° cases we will now show that the asymptotic forms are the exact forms for all i.

Consider

$$P^m_{n_i}(0) = \frac{\sqrt{\pi}\, 2^m}{\Gamma[\frac{1}{2}(n_i - m) + 1]\Gamma[\frac{1}{2} - \frac{1}{2}(n_i + m)]} \qquad \text{(Ref. 4, p. 63)}$$

Substituting $m = 1$, the case under discussion, one obtains

$$P'_{n_i}(0) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}n_i + \frac{1}{2})\Gamma(-\frac{1}{2}n_i)}. \tag{1.2}$$

Since $P^m_{n_i}(x) = P^m_{-n_i-1}(x)$ (Ref. 4, p. 62) in order not to introduce degenerate forms we will find the critical value of n_i and only consider n_i greater than this critical value.

Thus

$$n_i^c = -n_i^c - 1 \text{ i.e., } n_i^c = -\tfrac{1}{2}$$

Therefore, we will only consider those n_i's such that

$$n_i > -\tfrac{1}{2} \tag{1.3}$$

By (1.1) and (1.2) this would mean, when $\theta_0 = \pi/2$, that

$$\frac{1}{\Gamma(\frac{1}{2}n_i + \frac{1}{2})\Gamma(-\frac{1}{2}n_i)} = 0$$

Thus, either $\frac{1}{2}(n_i + 1)$ or $-\frac{1}{2}n_i$ must be poles of the gamma function. $\Gamma(\frac{1}{2}n_i + \frac{1}{2})$ has no poles for $n_i > -\frac{1}{2}$, but $\Gamma(-\frac{1}{2}n_i)$ has poles for $n_i = 0, 2, 4, 6, \cdots$. Therefore

$$n_i = 2i \qquad i \geq 0 \tag{1.4}$$

In the case of $\theta_0 = 180°$

$$P'_{n_i}(x) = -\tfrac{1}{2}n_i(n_i + 1)(1 - x^2)^{\frac{1}{2}}\,{}_2F_1[1 - n_i, n_i + 2; 2; \tfrac{1}{2}(1 - x)] \tag{1.5}$$

(Ref. 4, p. 63)

TABLE 1

θ_0	π/θ_0
90°	2.0000
165°	1.0909
170°	1.0588
175°	1.0286
180°	1.0000

We observe as $x \to -1$ that the right hand side of (1.5) becomes indeterminate; thus, to evaluate it at this point we must determine

$$\lim_{z=(\pi-\theta)\to+0} P'_{n_i}(\cos\theta) \tag{1.6}$$

$$= -\tfrac{1}{2}n_i(n_i + 1)z\,{}_2F_1(1 - n_i, n_i + 2; 2; [1 - \tfrac{1}{4}z^2 + 0(z^4)])$$

Since the ${}_2F_1$ diverges for $x = -1$ in (1.5), and since we can expand (1.6) in a Maclaurin series in z, and since z times it also diverges, then a priori, (1.6) diverges. In order that (1.6) be consistent with (1.1), ${}_2F_1[1 - n_i, n_i + 2; 2; \frac{1}{2}(1 - x)]$ must cut off. Hence, either $(1 - n_i)$ or $(n_i + 2)$ should be zero or a negative integer. By (1.3), $(n_i + 2)$ is always $> 3/2$; thus, $(1 - n_i)$ must equal zero or a negative integer, i.e.,

$$n_i = i \qquad i = 1, 2, 3, \cdots. \tag{1.7}$$

Thus, we observe that the asymptotic form for the differences (Table 1) is the exact form for all $(n_{i+1} - n_i)$ in the case of $\theta_0 = 90°$ and 180°.

A consequence of MacDonald's theorem, "as θ increases from 0 to π, any zero diminishes" (Ref. 5), is that the differences between successive zeros, for the same value of θ, decrease with increasing θ from 0 to π.

For $\theta_0 = 165°$ the maximum difference between the successive zeros is 1.0909 and the minimum difference is 1; for $\theta_0 = 170°$ the maximum difference is 1.0588.

2. The Method of Solution. Schelkunoff [6, p. 514] expanded the Legendre unction in a Taylor expansion

$$P_{n'+z_{n'}}(x) = P_{n'}(x) + 2z_{n'}[\tfrac{1}{2}P_{n'}(x)\log\tfrac{1}{2}(1+x) + S'_{n'}] \tag{2.1}$$

where (with $[n']$ = largest integer in n')

$$S'_{n'} = \sum_{s=1}^{[n']} \frac{(-1)^s(n'+s)!}{(s!)^2(n'-s)!}\left\{\frac{1}{n'+s}+\cdots+\cdot\cdot+\frac{1}{n'+1}\right\}\left(\frac{1-x}{2}\right)^s \tag{2.1a}$$

Another form for the S'_n is

$$S'_{n'} = (-1)^{[n']}\sum_{s=0}^{[n']}\frac{(-1)^s(n+s)!}{(s!)^2(n-s)!}\left\{\frac{1}{s+1}+\cdots+\frac{1}{s+n'}\right\}\left(\frac{1+x}{2}\right)^s \tag{2.1b}$$

Differentiating (2.1) with respect to x we obtain

$$\frac{dP_{n'+z_{n'}}(x)}{dx} = \frac{dP_{n'}(x)}{dx} + z_{n'}\left\{\frac{dP_{n'}(x)}{dx}\log\frac{1+x}{2} + \frac{P_{n'}(x)}{1+x} + 2\frac{dS'_{n'}}{dx}\right\} \tag{2.2}$$

Setting $x = x_0$ and remembering that $P'_{n_i}(x_0) = 0$ we obtain

$$z_{n'} = \left\{\frac{-\dfrac{dP_{n'}(x)}{dx}}{\dfrac{P_{n'}(x)}{1+x}+\dfrac{dP_{n'}(x)}{dx}\log\dfrac{1+x}{2}+2\dfrac{dS'_{n'}}{dx}}\right\}_{x=x_0} \tag{2.3}$$

where $n_i = n' + z_{n'}$

Equation (2.3) is a relationship between Legendre Functions and Associated Legendre Functions and a finite sum $dS'_{n'}/dx$. To make it numerically useful one must be able to evaluate the functions on the right without having to sum slowly convergent series.

If we consider n' either an integer or an odd multiple of $\frac{1}{2}$, we can evaluate the functions on the right of (2.3) quite easily. Some tables exist for the Legendre Polynomials and their derivatives (Ref. 7 and 8). When n' is considered an odd multiple of $\frac{1}{2}$, several methods of obtaining tables are available when one utilizes Hall's observation (Ref. 9) that the $\frac{1}{2}$ integer P's can be expressed in terms of the complete elliptic integrals, K and E, of modulus $\frac{1}{2}(1-x_0)$.

$$P_{-\frac{1}{2}}(x) = (2/\pi)K \qquad \text{(Ref. 9, p. 927)} \tag{2.4}$$
$$P_{\frac{1}{2}}(x) = (2/\pi)(2E-K)$$

These functions have been tabulated (Ref. 10). There are many ways to obtain the other $\frac{1}{2}$ integer Legendre and Associated Legendre Functions. One could use the recursion relationship between the Legendre and Associated Legendre Functions. In Section 4 these functions for $x_0 = \cos 165°$ are evaluated, and the existing tables were used as checks.

To improve our results we added a third term to the Taylor expansion (2.1) and thus instead of (2.3) we solved a resulting quadratic equation in $z_{n'}$.

Thus, to obtain the zeros of

$$P'_{n_i}(x_0) = [-\sqrt{1-x^2}(dP_{n_i}(x)/dx)]_{x=x_0} = 0 \tag{2.5}$$

we set $n_i = n' + z_n$, and compute two columns (one for $n' = j + \frac{1}{2}$ and one for $n' = j; j = 1, 2, 3, \cdots$), as shown in Table 2.

Table 2 is obtained from a Taylor expansion in z which was terminated after three terms, thus committing an error proportional to z cubed. The obvious criteria then is to choose a_k if $| z_k'' | < | z_k' |$ or a_k' if $| z_k'' | > | z_k' |$.

A method of predicting which part of Table 2 should be used, even before the n_i's are computed, is available for most of the n_i's for any given angle. We shall proceed to illustrate this for $x_0 = \cos 165°$.

TABLE 2

Zeros (n_i)	A $j + \frac{1}{2} \pm z_j''$	A' $j \pm z_j'$
n_1	a_1	a_1'
n_2	a_2	a_2'
.	.	.
.	.	.
.	.	.
n_k	a_k	a_k'
.	.	.
.	.	.
.	.	.

where $z_j'' = z_{j+\frac{1}{2}}$ & $z_j' = z_j$

In Reference 2 it was seen that the integer parts of the zeros could be determined from the Associated Legendre Functions. The example given in that reference was

$$\frac{1}{\sin 165°} P'_{13}(\cos 165°) = 6.50140$$

$$\frac{1}{\sin 165°} P'_{14}(\cos 165°) = -.86327 \tag{2.6}$$

$$\frac{1}{\sin 165°} P'_{15}(\cos 165°) = -5.23851$$

and

$$1.0321 > n_1 > 1.0316 \tag{2.7}$$

From (2.6) we note that a change in sign exists between $P'_{13}(\cos 165°)$ and $P'_{14}(\cos 165°)$ but not between $P'_{14}(\cos 165°)$ and $P'_{15}(\cos 165°)$. Thus, $P'_{n_i}(x_0) \neq 0$ for $14 \leqq n_i \leqq 15$. Since by (2.7) $n_1 = 1.032$ and since the largest difference is 1.0909, then each n_i for $1 \leqq i \leqq 13$ has the number i to the left of the decimal point. For $14 \leqq i \leqq N$, where N is at least 23, the n_i has the number $(i + 1)$ to the left of the decimal point.

Since we have observed maximum and minimum differences for cos 165°

(1.0909 and 1) we can obtain more information from the fact that no change in sign occurs between $P'_{14}(\cos 165°)$ and $P'_{15}(\cos 165°)$; namely,

$$\begin{aligned} &16.1818 > n_{15} > 16, \mid z'_{15} \mid < .1818 \\ &15.0909 > n_{14} > 15, \mid z'_{14} \mid < .0909 \\ &14 > n_{13} > 13.9091, \mid z'_{13} \mid < .0909 \\ &13 > n_{12} > 12.8182, \mid z'_{12} \mid < .1818 \end{aligned} \tag{2.8}$$

and also, from (2.7)

$$\begin{aligned} &1.0321 > n_1 > 1.0316 \\ &2.1230 > n_2 > 2.0316 \\ &3.2139 > n_3 > 3.0316 \end{aligned} \tag{2.9}$$

By (2.9), for at least the first three zeros, column A′ of Table 2 is better than column A. By (2.8), for n_{12}, n_{13}, n_{14}, and n_{15}, again column A′ is better. The same sort of reasoning can be applied to the $P'_{i+1}(x)$ to tell when column A of Table 2 should be used.

Since there is no change in sign between $P'_{7.5}(x_0)$ and $P'_{8.5}(x_0)$ and no change in sign between $P'_{19.5}(x_0)$ and $P'_{20.5}(x_0)$, it can be concluded that

$$\begin{aligned} &9.6818 > n_9 > 9.5, 8.5909 > n_8 > 8.5, \\ &7.5 > n_7 > 7.4091, 6.5 > n_6 > 6.3182 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &21.6818 > n_{20} > 21.5, 20.5909 > n_{19} > 20.5, \\ &19.5 > n_{18} > 19.4091, 18.5 > n_{17} > 18.3182. \end{aligned} \tag{2.11}$$

and thus for the zeros indicated in (2.10) and (2.11), the results of column A in Table 2 should be used.

Then there will be very few cases where one should average between the tabulated data of columns A and A′ in Table 2. In Section 4 in our numerical example we shall follow this type of reasoning and select either the value in column A or that in column A′. It could be pointed out that instead of using a table with two columns and thus creating a scientific selection process, one could use either column A or A′ and add more terms to the Taylor expansion. Not only would this add more work than the suggested method (and also create places of increased error) but also would remove redundancy from the system. Since these computations were carried out on hand calculating machines, whenever we have a choice between two methods of almost equal over-all accuracy we choose the method with the most redundancy, a valuable means of checking computations. In this method we always have two values of n_i although one of these values is discarded in the final result. The two values serve as a check inasmuch as we would immediately be alerted if the two values were very different.

The integrals $\int_1^{x_0} [P_{n_i}(x)]^2\,dx$, $\int_1^{x_0} [P'_{n_i}(x)]^2\,dx$, and $\int_1^{x_0} [P'_{m_i}(x)]^2\,dx$ have been

of much interest in our electromagnetic scattering work (Ref. 11). In Reference 11, it was shown that

$$\int_1^{x_0} [P'_{n_i}(x)]^2\,dx = n_i(n_i+1)\int_1^{x_0} [P_{n_i}(x)]^2\,dx \tag{2.12}$$

where the n_i are the zeros of $P'_{n_i}(x_0) = 0$. It was also shown that

$$\int_1^{x_0} [P_{n_i}(x)]^2\,dx = \frac{-(1-x_0^2)}{2n_i+1} P_{n_i}(x_0)\left[\frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=n_i,\ x=x_0} \tag{2.13}$$

In addition it was shown that when the m_i are zeros of $[dP'_{m_i}(x)/dx]_{x=x_0} = 0$ we obtain

$$\int_{x_0}^1 [P'_{m_i}(x)]^2\,dx = m_i(m_i+1)(x_0^2-1)P_{m_i}(x_0) \cdot\left[\frac{P_{m_i}(x)}{x} + \frac{m_i(m_i+1)}{x(2m_i+1)}\frac{\partial P_y(x)}{\partial y} + \frac{1}{2m_i+1}\frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=m_i,\ x=x_0} \tag{2.14}$$

With respect to these integrals we can employ the methods described in this section to the right hand members of (2.13) and (2.14). In Section 3 the requisite algebra for (2.13) is shown and in Section 4 the results of its application to the case $x_0 = \cos 165°$ are given.

One might observe that the algebra required to put (2.14) in a similar computational form would be no more difficult, although it would be more lengthy.

3. The Computational Forms. In Section 2 the method of approach used in finding the a_k and a'_k of Table 2 was described; in this section we display the details of the computational method.

For column A′ of Table 2 we start with

$$P_{j+z'_j}(x) = \left[P_y(x) + z'_j\frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2}z'^2_j\frac{\partial^2 P_y(x)}{\partial y^2}\right]_{y=j} \tag{3.1a}$$

and for column A of the table we start with

$$P_{j+\frac{1}{2}+z''_j}(x) = \left[P_y(x) + z''_j\frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2}z''^2_j\frac{\partial^2 P_y(x)}{\partial y^2}\right]_{y=j+\frac{1}{2}} \tag{3.1b}$$

Differentiating these equations with respect to x, evaluating at $x = x_0$, and under the stipulation that $n_i = j + z'_j$ in (3.1a) and $n_i = j + \frac{1}{2} + z''_j$ in (3.1b), we obtain the quadratic equations referred to in Section 2. Solving them for z_j we obtain, in the first case

$$z'_j = \left[\left\{\frac{\partial^2 P_y(x)}{\partial x \partial y^2}\right\}^{-1} \cdot\left\{-\frac{\partial^2 P_y(x)}{\partial x \partial y} \pm \sqrt{\left(\frac{\partial^2 P_y(x)}{\partial x \partial y}\right)^2 - 2\left(\frac{\partial P_y(x)}{\partial x}\right)\left(\frac{\partial^2 P_y(x)}{\partial x \partial y^2}\right)}\right\}\right]_{y=j,\ x=x_0} \tag{3.2a}$$

and in the second case

$$z_j'' = \left[\left\{\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right\}^{-1} \cdot \left\{-\frac{\partial^2 P_y(x)}{\partial x \partial y} \pm \sqrt{\left(\frac{\partial^2 P_y(x)}{\partial x \partial y}\right)^2 - 2\left(\frac{\partial P_y(x)}{\partial x}\right)\left(\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right)}\right\}\right]_{y=j+\frac{1}{2},\ x=x_0} \tag{3.2b}$$

The expressions that were computed in order to find the z_j associated with a given j or $j + \frac{1}{2}$ are those shown in (3.2a) and (3.2b). However, before we comment on them, let us turn our attention to the integral, referred to in Section 2,

$$\int_{x_0}^{1} [P_{n_i}(x)]^2\, dx = \frac{1 - x_0^2}{2n_i + 1} P_{n_i}(x_0) \left[\frac{\partial^2 P_y(x)}{\partial x \partial y}\right]_{y=n_i,\ x=x_0} \tag{3.3}$$

In order to find both the n_i's and the values of these integrals we observe that in addition to the forms required in the right hand members of (3.2a) and (3.2b) we will need $P_{n_i}(x_0)$ and $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_i,\ x=x_0}$. After the z_j's have been obtained, the $P_{n_i}(x_0)$ are determined through (3.1a) and (3.1b). As for $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_i,\ x=x_0}$, we consider

$$\frac{\partial P_y(x)}{\partial y}\bigg|_{y=n_j+z_j} = \left[\frac{\partial P_y(x)}{\partial y} + z_j \frac{\partial^2 P_y(x)}{\partial y^2} + \tfrac{1}{2} z_j^2 \frac{\partial^3 P_y(x)}{\partial y^3} + \cdots\right]_{y=n_j} \tag{3.4}$$

and

$$P_{n_j + z_j}(x) = \left[Py(x) + z_j \frac{\partial P_y(x)}{\partial y} + \tfrac{1}{2} z_j^2 \frac{\partial^2 P_y(x)}{\partial y^2} + \cdots\right]_{y=n_j} \tag{3.4a}$$

where $n_j = j$ in the one case and $n_j = j + \frac{1}{2}$ in the other.

Combining these two equations, differentiating with respect to x, and then solving for $[\partial^2 P_y(x)/\partial x \partial y]_{y=n_j+z_j}$ we obtain, with the boundary condition $P'_{n_i}(x_0) = 0$ which is equivalent to $\partial P_{n_i}(x)/\partial x \mid_{x=x_0} = 0$,

$$\frac{\partial^2 P_y(x)}{\partial x \partial y}\bigg|_{y=n_j+z_j,\ x=x_0} \cong \left\{-\frac{1}{z}\frac{\partial P_{n_j}(x)}{\partial x} + \frac{z_j}{2}\frac{\partial^3 P_y(x)}{\partial x \partial y^2}\right\}_{y=n_j,\ x=x_0} \tag{3.5}$$

Now examination of (3.1a), (3.1b), (3.2a), (3.2b) and (3.5) shows what quantities require computation. For both the $j + z_j'$ and $j + \frac{1}{2} + z_j''$ methods, we start with the form

$$P_y(x) = \frac{\sin y\pi}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!} \cdot \left\{\log\frac{1+x}{2} + \psi(y+s) + \psi(y-s) - 2\psi(s)\right\}\left(\frac{1+x}{2}\right)^s + \cos y\pi \sum_{s=0}^{\infty} \frac{(-1)^s (y+s)!}{(s!)^2 (y-s)!}\left(\frac{1+x}{2}\right)^s \tag{3.6}$$

(Ref. 12, p. 54)

which is transformed by the use of the relation

$$\psi(-z) = \psi(z-1) + \pi \cot \pi z \tag{3.7}$$

(Ref. 13, p. 19)

into

$$
\begin{aligned}
P_y(x) = {} & \frac{\sin y\pi}{\pi} \sum_{s=0}^{j} \frac{(-1)^s(y+s)!}{(s!)^2(y-s)!} \\
& \cdot \left\{ \log \frac{1+x}{2} + \psi(y+s) + \psi(y-s) - 2\psi(s) \right\} \left(\frac{1+x}{2} \right)^s \\
& + \cos y\pi \sum_{s=0}^{j} \frac{(-1)^s(y+s)!}{(s!)^2(y-s)!} \left(\frac{1+x}{2} \right)^s \\
& + \frac{\sin y\pi}{\pi} \sum_{s'=0}^{\infty} \frac{(-1)^{s'+j+1}}{[(s'+j+1)!]^2} \frac{(y+s'+j+1)!}{(y-j-1-s')!} \\
& \cdot \left\{ \log \frac{1+x}{2} + \psi(y+s'+j+1) + \psi(j+s'-y) - 2\psi(s'+j+1) \right\} \\
& \cdot \left(\frac{1+x}{2} \right)^{s'+j+1}
\end{aligned}
\tag{3.8}
$$

Starting with (3.8) each of the derivatives required is determined and approximated by taking a finite number of terms in the corresponding series. The number of terms one would use would depend upon the number of significant digits required. It was found that the first $(j+1)$ terms of each series would would give sufficient accuracy for our purposes. For example in finding $[\partial P_y(x)/\partial y]_{y=j+\frac{1}{2},\, x=x_0}$, when $j=1$ the error committed by restricting the sum to the first $(j+1)$ terms is less than .009 while for $j=3$ the error is less than 3×10^{-6}.

With forms determined for the derivatives, the equations (3.2a) and (3.2b) could then be employed to determine the values of z_j'' and z_j'. Then the values of the zeros are determined by

$$
\begin{aligned}
n_i &= z_j' + j \qquad \text{where } i = j \text{ for } 1 \leqq i \leqq 13 \\
&\qquad\qquad\quad\; i = j - 1 \text{ for } 14 \leqq i \leqq 23
\end{aligned}
\tag{3.9a}
$$

and

$$
n_i = j + \tfrac{1}{2} + z_j'' \tag{3.9b}
$$

To determine the values of the integrals both the derivatives and the Legendre and the Associated Legendre Functions of integer degree are required. These Legendre and Associated Legendre Functions of integer degree were expressed in polynomial form so that their values could be determined by substitution. With these determined, the integrals are placed in computational form (see (3.5) and (3.3)).

The results that one obtains when this scheme is used are illustrated by the example which is discussed in the following section.

4. Results for $x_0 = \cos 165°$. The values for Legendre and Associated Legendre Functions of integer and half-integer degree are tabulated in Table 3.

Table 4 contains the results of our computation.

TABLE 3

Values of Legendre Functions for $x_0 = \cos 165$

ν	$P_\nu(x_0)$	$-P'_\nu(x_0)/\sqrt{1-x_0^2}$	ν	$P_\nu(x_0)$	$-P'_\nu(x_0)/\sqrt{1-x_0^2}$
1	−.96592 58263	+1.00	−.5	2.1854 43649 8	
2	+.89951 90528	−2.8977 77479	.5	−.88034 27550	
3	−.80416 39229	+5.4975 95264	1.5	.40531 3187	−10.946 2606
4	+.68469 54388	−8.5269 24940	2.5	−.098198 308	11.586 5587
5	−.54712 58747	+11.659 85421	3.5	−.12690 551	−11.535 451
6	+.39830 59910	−14.545 30956	4.5	.29429 8791	10.571 315
7	−.24554 10452	+16.837 83210	5.5	−.41302 4224	−8.5924 6265
8	+.09618 43272 4	−18.228 42524	6.5	.48750 2435	5.6150 241
9	+.04276 78470 9	+18.472 96566	7.5	−.52104 2564	−1.7674 284
10	−.16505 59738	−17.415 83615	8.5	.51721 7324	−2.7216 569
11	+.26548 99921	+15.006 79021	9.5	−.48040 2367	7.5424 833
12	−.34021 56675	−11.309 56633	10.5	.41591 3951	−12.329 704
13	+.38689 98145	+6.5013 98521	11.5	−.32992 1717	16.692 590
14	−.40482 44602	−.86327 13353	12.5	.22922 4586	−20.247 8253
15	+.39488 56155	−5.2385 10824	13.5	−.12094 3791	22.652 4294
16	−.35949 81090	+11.378 18274	14.5	.012172 729	−23.634 2515
17	+.30241 35834	−17.101 94842	15.5	.090383 6368	23.017 611
18	−.22846 39061	+21.962 65816	16.5	−.18075 1621	−20.741 975
19	+.14324 66015	−25.555 11295	17.5	.25398 9737	16.872 056
20	−.05277 21934 5	+27.549 27562	18.5	−.30642 7855	−11.598 345
			19.5	.33582 976	5.2278 00
			20.5	−.34146 94	1.8349

TABLE 4

Results for the case $x_0 = \cos 165°$

$n_i = j + s'_j$	$\int_{x_0}^{1} [P_{n_i}(x)]^2\,dx$	$n_i = j + \frac{1}{2} + s''_j$	$\int_{x_0}^{1} [P_{n_i}(x)]^2\,dx$
1.03158 ✓	.62837 ✓	1.08929	.98926
2.08361 ✓	.37588 ✓	2.13167	.56654
3.14588 ✓	.27853 ✓	3.18431	.36562
4.21190 ✓	.23211 ✓	4.24367	.25318
5.27900	.20834	5.31014 ✓	.18467 ✓
6.34761	.19719	6.38439 ✓	.14195 ✓
7.42048	.19522	7.46564 ✓	.11637 ✓
8.60072	.17093	8.55028 ✓	.10286 ✓
9.67183	.13271	9.63336 ✓	.09742 ✓
10.74310	.10509	10.71202 ✓	.09686 ✓
11.81852 ✓	.08513 ✓	11.78681	.09933
12.90018 ✓	.07148 ✓	12.86075	.10428
13.98719 ✓	.06345 ✓	13.93859	.11224
15.07534 ✓	.06034 ✓	15.12111	.09485
16.15973 ✓	.06085 ✓	16.19451	.07691
17.23867 ⊗	.06368 ⊗	17.26897 ⊗	.06338 ⊗
18.31399	.06816	18.34815 ✓	.05344 ✓
19.38967	.07433	19.43351 ✓	.04689 ✓
20.47144	.08297	20.52279 ✓	.04374 ✓

✓ indicates preferred value (according to criterion of Section #2).
⊗ indicates need for averaging (according to criterion of Section #2).

In conclusion it should be pointed out that no effort has been made to smooth out results as the major virtue of this method is the convenience and speed of computation.

The series for the Legendre Functions involved are very slowly convergent and any effort made to "reinforce" unreliable entries in Table 4 would imply the knowledge of a correct entry (the calculation of which we wish to by-pass in obtaining answers to physical problems).

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CONVERGENCE OF EXPANSIONS IN THE HERMITE POLYNOMIALS $H_n(hw)$

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1. Introduction. The present paper is devoted to a discussion of the effect of a complex parameter h on the convergence of a series in Hermite polynomials $H_n(hw)$. The conditions for the convergence of a given series

$$f(z) = \sum_{n=0}^{\infty} f_n H_n(z), \qquad z = x + iy, \tag{1}$$

where the $H_n(z)$ are the Hermite polynomials of degree n defined by[1]

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi}\, 2^n n! \delta_n^m \tag{2}$$

have been discussed by a number of writers, most complete results having been obtained by Hille,[2] who gave an expression for the width 2τ of the strip in the z plane in which such a series converges. In case $\tau > 0$, inversion of the series for the coefficients f_n gives

$$f_n = \frac{1}{\sqrt{\pi}\, 2^n n!} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x)\, dx. \tag{3}$$

In a method used in statistics for the analysis of frequency distributions, it is customary to represent a given distribution function $g(x)$ by a Gram-Charlier series

$$g(x) = e^{-h^2x^2} \sum_{n=0}^{\infty} g_n H_n(hx) \tag{4}$$

where g_1 may be made to vanish by choosing the mean value of x for the origin. Choice of $h = 1/\sqrt{2}\sigma$, where σ is the standard deviation of the distribution, then makes g_2 vanish. It is interesting to note that an analogous method of choosing the parameter appearing in the exponential was used by Boltzmann[3] in his expansion of the distribution function in velocity space. Boltzmann used a multiple power series in the velocity coordinates u, v and w in place of a series of Hermite polynomials, and chose the parameter so as to make the sum of the coefficients of the u^2, v^2 and w^2 terms vanish. This suggests the mathematically interesting question as to the effect of the choice of the parameter on the convergence of such expansions, and as to the establishment of a criterion for the selection of an optimum value of the parameter. This question also arises in the expansion

$$\psi(x) = e^{-h^2x^2/2} \sum_{n=0}^{\infty} h_n H_n(hx) \tag{5}$$

[1] G. Szegö, *Orthogonal Polynomials*, Am. Math. Soc., 101–102 (1939).

[2] E. Hille, Duke Math. Jour., 5, 875–936 (1939).

[3] L. Boltzmann, *Vorlesungen über Gastheorie*, Leipzig, Johann A. Barth, I Theil, 184–190 (1896).

for the wave function of a one-dimensional quantum-mechanical harmonic oscillator. In this case $\underline{h}$ is determined by Planck's constant and by the mass and the potential function of the oscillator, and this pre-determined value is not expected to be the optimum value for a given wave function.

2. General Theory. Let us consider the formal expansion of a function $g(w)$ of a complex variable w in the form

$$g(w) = e^{-\underline{k}\underline{h}^2w^2} \sum_{n=0}^{\infty} g_n H_n(\underline{h}w), \tag{6}$$

$\underline{k}$ being a pre-determined constant and $\underline{h}$ a complex parameter.

If, in (2), we make the substitution $z = \underline{h}w$, we obtain

$$\int_{-\infty/\underline{h}}^{\infty/\underline{h}} e^{-\underline{k}\underline{h}^2w^2} H_n(\underline{h}w) H_m(\underline{h}w)\, dw = \sqrt{\pi}\, 2^n n! \underline{h}^{-1} \delta_n^m, \tag{7}$$

where the limits indicate that the integration is along the line in the w-plane into which the real axis of the z-plane is transformed by the substitution. We thus obtain for the coefficients g_n in the expansion (6):

$$g_n = (\underline{h}/\sqrt{\pi}\, 2^n n!) \int_{-\infty/\underline{h}}^{\infty/\underline{h}} e^{(\underline{k}-1)\underline{h}^2w^2} g(w) H_n(\underline{h}w)\, dw. \tag{8}$$

The convergence properties of a series of Hermite polynomials, such as given by (1), have been discussed by Hille,[4] who proved that the region of convergence of (1) is a strip $-\tau < y < \tau$, with the possible addition of points on the lines $y = \pm\tau$, where

$$\tau = -\limsup_{n\to\infty} (1/\nu_n) \log (A_n \mid f_n \mid) \tag{9}$$

in which A_n is defined by

$$A_{2m} = \mid H_{2m}(0) \mid = \frac{(2m)!}{m!}, \qquad A_{2m+1} = \frac{\mid H'_{2m+1}(0) \mid}{(4m+3)^{\frac{1}{2}}} = \frac{2(m+1)!}{m!(4m+3)^{\frac{1}{2}}} \tag{10}$$

and $\nu_n = (2n+1)^{\frac{1}{2}}$. Hille further noted that for A_n and ν_n we may use the values

$$A_n = 2^n \Gamma(\tfrac{1}{2}n + a), \qquad \nu_n = \sqrt{2n} \tag{10'}$$

where a is any arbitrary but fixed real number.

Writing $\underline{h} = \mid \underline{h} \mid \exp(i\theta)$, $w = (z/\underline{h}) = (z/\mid \underline{h} \mid) \exp(-i\theta)$, $0 \leqq \theta < 2\pi$, we note that the series given by (6) and (8) converges in the strip in the w-plane which is the image of the strip in the z-plane bounded by the lines $y = \pm\tau$. Since we may write (8) as

$$g_n = (1/\sqrt{\pi}\, 2^n n!) \int_{-\infty}^{\infty} e^{(\underline{k}-1)x^2} g(x/\underline{h}) H_n(x)\, dx, \tag{11}$$

we may apply Hille's result to find the width of the strip in the w-plane for the series (6). Thus, there exists a strip in the w-plane, in which the series (6) converges, if and only if $\underline{h}$ has a value such that

$$\tau = -\limsup_{n\to\infty} 1/\sqrt{2n} \log [2^n \Gamma(\tfrac{1}{2}n + a) \mid g_n \mid] > 0. \tag{12}$$

[4] Op. cit., pp. 884–885.

We conclude that, in general, the equation

$$\limsup_{n\to\infty} (1/\nu_n) \log (A_n \mid g_n \mid) = 0 \tag{13}$$

defines a curve in the h-plane which divides this plane into two regions: C, the region where $\tau > 0$, giving values of $\underline{h}$ for which there exists a strip of convergence in the w-plane, and D, the region where $\tau < 0$, giving values of $\underline{h}$ for which there exists no strip of convergence in the w-plane. It is to be noted that the regions C and D may be multiply-connected.

In case the sequence

$$\gamma_n = -(1/\nu_n) \log (A_n \mid f_n \mid), \qquad n = 0, 1, 2, \cdots \tag{14}$$

is bounded, an optimum value of $\underline{h}$ may be defined as the value of $\underline{h}$ giving the maximum value of τ. If this sequence is not bounded but contains a monotone increasing subsequence τ_{n_i} such that $\lim_{i\to\infty} \tau_{n_i} = +\infty$, it may happen that

$$\lim_{i\to\infty} (\tau_{n_i}/n^r) = K(h) \tag{15}$$

where $K(h)$ is not identically 0, for some greatest number $\underline{r}$. Then the value of $\underline{h}$ which makes $K(h)$ a maximum for all such subsequences may be taken as the optimum value of $\underline{h}$.

It is to be noted that the region in the h-plane and the function $K(h)$ remain unchanged when $g(w)$ is replaced by $g(w)P(w)$, where $P(w)$ is any entire function of order less than two, or of minimal type if of order two. In order to prove this, we note that, by Schwartz' inequality,

$$\left| \int_{-\infty}^{\infty} e^{(k-1)x^2} g(x/h)P(x/h)H_n(x)\, dx \right| \tag{16}$$

$$= S_n M \left\{ \int_{-\infty}^{\infty} \mid e^{(k-1+\delta)x^2} g(x/h)H_n(x) \mid^2 dx \right\}^{\frac{1}{2}}$$

where δ is an arbitrarily small positive number,

$$M = \left\{ \int_{-\infty}^{\infty} e^{-2\delta x^2} \mid P(x/h) \mid^2 dx \right\}^{\frac{1}{2}}$$

is a finite number not depending on n, and $0 \leqq S_n \leqq 1$. Thus, since

$$\limsup_{n\to\infty} (1/\nu_n) \log (S_n M) = 0,$$

the presence of the factor $P(w)$ does not affect the expressions obtained for τ and for $K(h)$.

Hille[5] has also obtained necessary and sufficient conditions which a function $f(z)$ must satisfy in order that an expansion of the type (5), with $h = 1$, shall exist and converge to the sum $f(z)$ in the strip of width 2τ in the z-plane. Hille's result may be readily extended to expansions of the type (6) by replacing $f(z)$ by $g(z/h) \cdot \exp(k - \frac{1}{2})z^2$ in the condition obtained by him.

[5] Hille, Trans. Am. Math. Soc., 47, 80–94 (1940).

3. Example. In order to determine the condition imposed on h by the inequality (12) for any entire function of order two, we need only consider the expansion of the function $g(w) = \exp(-\alpha w^2)$. The integral (11) for the coefficients g_n in the formal expansion (6) may be readily evaluated by means of the generating function[1] for the Hermite polynomials. The result is

$$g_{2m+1} = 0, \qquad g_{2m} = \frac{h(h^2k - \alpha)^m}{m!2^{2m}(\alpha - h^2k + h^2)^{m+1}}. \tag{17}$$

Using $a = 1$, we find that the expression (12) for τ may be reduced to

$$\tau = -\limsup_{m\to\infty} \frac{\sqrt{m}}{2} \log \left| \frac{h^2k - \alpha}{h^2k - h^2 - \alpha} \right|. \tag{18}$$

The condition for the existence of a strip of convergence in the w-plane is evidently

$$|h^2k - \alpha| < |h^2k - h^2 - \alpha| \tag{19}$$

which may be expressed more simply as

$$R[k - 1 - (\alpha/h^2)] < -\tfrac{1}{2}. \tag{20}$$

The boundary between the regions C and D is given by using an equality in place of the inequality. It is seen that the boundary in the h-plane maps into a curve in the h^2-plane which is the locus of points the ratio of whose distances from the points α/k and $\alpha/(k-1)$ is $|(1-k)/k|$. This curve in the h^2-plane is a circle which degenerates into a straight line when $k = \tfrac{1}{2}$.

In case $k = 0$, corresponding to expansions of the form (1), the boundary in the h^2-plane is a circle with center at $h^2 = -\alpha$ and radius $|\alpha|$, and $\tau = +\infty$ if h^2 is exterior to this circle. In case $k = \tfrac{1}{2}$, corresponding to (5), the boundary in the h^2-plane is the perpendicular bisector of the line segment connecting the points $h^2 = \alpha$ and $h^2 = -\alpha$, and $\tau = +\infty$ if h^2 is in the half-plane containing the point $h^2 = \alpha$. When $k = 1$, corresponding to (4), the boundary in the h^2-plane is a circle with center at $h^2 = \alpha$ and radius $|\alpha|$, and $\tau = +\infty$ if h^2 is interior to this circle. Thus expansions of the type (5) for $\psi(x) = P(x)\cdot\exp(-\alpha x^2)$ converge for any α such that $R(\alpha) > 0$, for any predetermined positive value of h.

To determine the optimum value of h, we note from (18) that

$$K(h) = -\tfrac{1}{2} \log \left| \frac{h^2k - \alpha}{h^2k - h^2 - \alpha} \right| \tag{21}$$

for $\tau = \tfrac{1}{2}$ in (15). Thus the optimum values of h are $h = \infty, \sqrt{\alpha}$ and $\sqrt{2\alpha}$ for $k = 0$, 1 and $\tfrac{1}{2}$, respectively, giving in each case $\max K(h) = \infty$.

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UPPER AND LOWER BOUNDS FOR THE NUMERICAL SOLUTION OF THE DIRICHLET DIFFERENCE BOUNDARY VALUE PROBLEM*

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1. Introduction. The purpose of the present note is to indicate a method for the numerical solution of the Dirichlet difference boundary value problem, which has the property that upper and lower bounds for the solution are obtained. The labor involved in obtaining upper and lower bounds appears to be not much more than the labor involved in applying more usual procedures of numerical solution, which, however, furnish no concrete estimate of the error in the final numerical approximation to the solution. Although the underlying idea of the present method is well known, the possibility of its systematic application for numerical purposes seemingly has not been noticed before.

Consider the Dirichlet difference boundary value problem on a finite set $D + C$ of points (m, n) in the xy-plane (for the precise formulation of the problem and other relevant information, see, for example, Diaz and Roberts [2]†). Given a real valued function f, defined on the "boundary" C, it is required to find a real valued function u, defined on $D + C$, such that u coincides with f on C, i.e.

$$u(m, n) = f(m, n), \tag{1}$$

at each point (m, n) of C, and that u satisfies the Laplace difference equation

$$4\,u(m, n) = u(m + 1, n) + u(m - 1, n) + u(m, n + 1) + u(m, n - 1), \tag{2}$$

at each point (m, n) of the "interior" D. This problem is well known to possess one and only one solution.

A real valued function v defined on $D + C$ is called superharmonic provided that $4v(m, n) \geqq v(m + 1, n) + v(m - 1, n) + v(m, n + 1) + v(m, n - 1)$, for each (m, n) in D. It is well known that the minimum over $D + C$ of a superharmonic function occurs on the boundary C, i.e. if (m, n) is in D then

$$v(m, n) \geqq \min_C v.$$

Given a "boundary value function" f defined on C, a superharmonic function v on $D + C$ will be called an "upper function for f" provided that, in addition to being superharmonic, v satisfies

$$v(m, n) = f(m, n),$$

for (m, n) on C. Let u be the solution (see (1) and (2)) of the Dirichlet problem with boundary values f, and v be any upper function for f. The superharmonic function $v - u$ vanishes on C, and its minimum over $D + C$ occurs on C. Therefore

$$u(m, n) \leqq v(m, n),$$

for any (m, n) in D.

* This work was sponsored by the Office of Naval Research.

† Numbers in brackets refer to the bibliography at the end of the paper.

A real valued function w defined on $D + C$ is called subharmonic provided that

$$4w(m, n) \leqq w(m + 1, n) + w(m - 1, n) + w(m, n + 1) + w(m, n - 1),$$

for each (m, n) in D. It is well known that the maximum over $D + C$ of a subharmonic function occurs on the boundary C, i.e. if (m, n) is in D then

$$\max_C w \geqq w(m, n).$$

Given a boundary value function f on C, a subharmonic function w on $D + C$ will be called a "lower function for f" provided that, in addition to being subharmonic, w satisfies

$$w(m, n) = f(m, n),$$

for (m, n) on C. Let u be the solution (see (1) and (2)) of the Dirichlet problem with boundary values f, and w be any lower function for f. The subharmonic function $w - u$ vanishes on C, and its maximum over $D + C$ occurs on C. Therefore

$$w(m, n) \leqq u(m, n),$$

for any (m, n) in D.

Thus, given f, and two functions v and w, the first an upper function for f and the second a lower function for f, one has, automatically, upper and lower bounds for the solution u:

$$w(m, n) \leqq u(m, n) \leqq v(m, n), \tag{3}$$

at any point (m, n) of D. This simple inequality is the basis for the following numerical method: Given f, start with any upper function v, which automatically furnishes a ("first") upper bound for the solution u at any point of D. (Such a function v can usually be easily constructed and in any case one may always start by taking for v the function which coincides with f at each point of C and whose numerical value at each point of D is defined to be the maximum value of f on C). Let (m, n) denote a fixed point of D and seek to improve the upper bound, $v(m, n)$, for the unknown number, $u(m, n)$, which is furnished by (3). This may be done by defining a new upper function, call it v_1, which coincides with v on $D - (m, n)$, and such that $v_1(m, n) \leqq v(m, n)$. As long as the number $v_1(m, n)$ is less than or equal to $v(m, n)$, the function v_1 will be superharmonic on $D - (m, n)$. But, in order for v_1 to be superharmonic at (m, n), one must have

$$\begin{aligned} 4v_1(m, n) &\geqq v_1(m + 1, n) + v_1(m - 1, n) + v_1(m, n + 1) + v_1(m, n - 1) \\ &= v(m + 1, n) + v(m - 1, n) + v(m, n + 1) + v(m, n - 1), \end{aligned}$$

so that the best possible value for the number $v_1(m, n)$ is precisely the "average":

$$\frac{v(m + 1, n) + v(m - 1, n) + v(m, n + 1) + v(m, n - 1)}{4}.$$

That is to say, if one starts with an arbitrary upper function for f, and one seeks to improve the upper bound for u given by (3), the best result at each point is obtained by applying the Liebmann [4] method (cf. method I of Diaz and Roberts [2]).

A similar discussion applies to the improvement of the lower bounds for u given by an arbitrary subharmonic function w in (3).

2. Example. An efficient method for making use of the general procedure sketched above may best be illustrated by means of a simple example. Consider the set of equally spaced points $D + C$ shown in Figure 1. This set con-

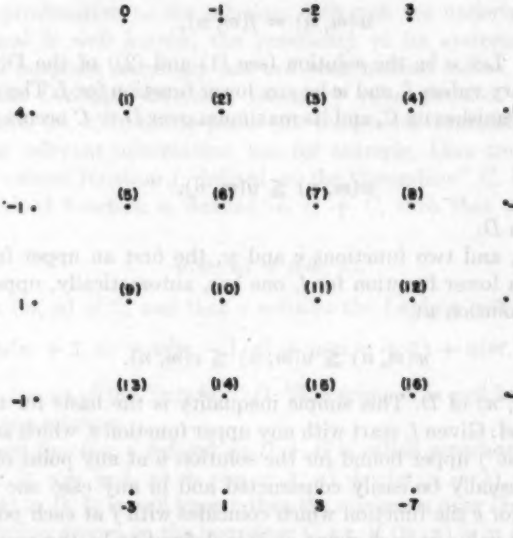

FIGURE 1

sists of sixteen interior points D and sixteen boundary points C. The boundary value problem consists in finding a function $u(m, n)$ which satisfies Laplace's difference equation in D and assumes the given boundary values f on C' (see Figure 1, where the numbers in parentheses associated with the interior points merely serve to denote the various points of D).

From the maximum-minimum principle it follows that

$$-7 \leqq u(m, n) \leqq 5,$$

for (m, n) in D. The function v, where $v(m, n) = 5$ for $m, n)$ in D, and $v(m, n) = f(m, n)$ for (m, n) on C, is an upper function for f. The function w, where $w(m, n) = -7$ for (m, n) in D, and $w(m, n) = f(m, n)$ for (m, n) on C, is a lower function

for f. The functions v and w may be used as initial functions for calculating approximately the solution u, as explained in the introduction, obtaining at the same time an estimate of the error.

If the functions v and w thus defined were used as initial superharmonic and subharmonic functions, one could then proceed towards a solution by one of the methods previously discussed (see Diaz and Roberts [2]), e.g. the Liebmann procedure, moving from point to point in D in the order given by Figure 1. This procedure, however, was found to converge rather slowly, and it was found expedient to calculate first better initial functions v_1 and w_1 before applying systematically the Liebmann procedure.

More desirable functions v_1 and w_1 may be found by using what may be called the relaxation method (Southwell [7]), in the following way. Using the functions v and w one computes their "residuals" in D. (By the residual of a function v is meant, as usual, its Laplacian Δv:

$$v(m, n) = [v(m + 1, n) + v(m - 1, n) + v(m, n + 1) + v(m, n - 1)] - 4v(m, n).$$

One then "relaxes" by making suitable unit or multiple unit changes (decreases for v and increases for w) in the values of the functions v and w in D. To make this explicit, suppose v_* is an upper function for f, and v_1^* is also an upper function for f, obtained from v_* by redefining v_* only at the point (m, n). Then, if

$$v_1^*(m, n) = v_*(m, n) - 1,$$

then Δv_* will coincide with Δv_1^* save at the five points: (m, n) and its four neighbors $(m + 1, n)$, $(m - 1, n)$, $(m, n + 1)$, and $(m, n - 1)$; and

$$\Delta v_1^*(m, n) = \Delta v_*(m, n) + 4,$$

$$\Delta v_1^*(i, j) = \Delta v_*(i, j) - 1,$$

where (i, j) is any of the four points $(m + 1, n)$, $(m - 1, n)$, $(m, n + 1)$, or $(m, n - 1)$. This relaxation process is repeated until it is impossible to relax any further ("by unit steps") without changing the sign of the residuals at some point. It must be emphasized that any "overshooting" (i.e., changing the sign of the residuals) during the relaxation destroys the sub and super harmonic properties of the functions involved, and hence also destroys the assurance that upper and lower bounds for the solution will be obtained.

The functions resulting from the above procedure are the initial improved functions v_1 and w_1. They have the additional property that their residuals are, roughly speaking, of the same magnitude but opposite in sign. This indicates that a reasonable approximation to the exact solution may be found by taking the arithmetic mean $\frac{1}{2}(v_1 + w_1)$. The results obtained by carrying out the indicated steps in the case of the problem in Figure 1 are shown in Table 1.

The Liebmann procedure may now be applied to the functions v_1 and w_1 to reduce the size of the error. Although this process does not converge rapidly,

the initial functions v_1 and w_1 are now so adjusted that increasingly better approximations to the exact solution may be expected by taking the arithmetic mean at appropriate stages, when the residuals are again roughly of the same magnitude but opposite in sign. Various modifications of the basic procedure outlined here will readily suggest themselves to a computer, depending on the available time and on the computing facilities at his disposal.

3. Concluding remarks. In a previous note (Diaz and Roberts [2]) the proof of the convergence of three iterative methods for the numerical solution of the difference equation boundary value problem which is analogous to the classical Dirichlet problem for Laplace's differential equation was given. These three methods are modelled after certain methods of solution for the Dirichlet differential equation problem: (a) Poincaré's [6] method of "sweeping out", (b) Kellogg's [3] extension of Poincaré's method, and (c) the fact that, intuitively speaking, "the steady state temperature is a harmonic function". In the previous note systematic use was made of super and sub harmonic functions in the case of the Laplace difference equation, these functions being the direct analogue of the super and sub harmonic functions introduced in the theory of Laplace's differential equation by Perron [5]. As a matter of fact, it is clear from the preceding considerations that the numerical method suggested in the present paper is essentially analogous to Perron's method of sub and super harmonic functions. To be exact, the present method rests on the precise analogue, for the Laplace difference equation, of the theorem of Perron [5] for the Laplace differential equation which relates the function which is the greatest lower bound

TABLE I

Point Number	Upper Bound v_1		Lower Bound w_1		Absolute Max. Error	Arithmetic Mean	Exact Solution
	Value	Residual	Value	Residual			
1	3	−2	−1	2	4	1	1
2	3	−3	−3	3	6	0	0
3	3	−3	−3	3	6	0	0
4	4	−2	0	2	4	2	2
5	3	−3	−3	3	6	0	0
6	4	−2	−4	2	8	0	0
7	4	−2	−4	2	8	0	0
8	3	−3	−3	3	6	0	0
9	3	−3	−3	3	6	0	0
10	4	−2	−4	3	8	0	0
11	4	−2	−4	3	8	0	0
12	3	−3	−3	3	6	0	0
13	1	−2	−3	3	4	−1	−1
14	3	−3	−2	0	5	0.5	0
15	3	−3	−2	0	5	0.5	0
16	3	−3	−2	0	5	0.5	0

of all superharmonic functions having the same prescribed boundary values (and also the least upper bound of all subharmonic functions having the same boundary values) to the solution (with the same boundary values) of the Dirichlet problem for Laplace's differential equation. Perron's theorem associates, to a given prescribed boundary value function f, two harmonic functions, one of which, call it $\bar{u}$, is the greatest lower bound of all superharmonic functions coinciding with (or not less than) f on the boundary, while the other harmonic function, call it $\underline{u}$, is the least upper bound of all subharmonic functions coinciding with (or not greater than) f on the boundary. Given a superharmonic function v and a subharmonic function of each of these classes, one has

$$w \leqq \underline{u} \leqq \bar{u} \leqq v,$$

but it cannot be asserted, in general, that the harmonic functions $\underline{u}$ and $\bar{u}$ "assume" the prescribed boundary values f. However, if some restrictive assumptions about the boundary of the domain are made, then it can be shown that indeed $\underline{u} = \bar{u}$ and that the harmonic function so obtained assumes the given boundary values f and is the solution of the Dirichlet problem. Thus, in the case of the Laplace differential equation the solution of the Dirichlet problem appears in two stages, so to speak. The first involves the construction of the harmonic functions $\underline{u}$ and $\bar{u}$, and the second, which depends upon certain smoothness assumptions concerning the boundary, consists in showing that $\underline{u} = \bar{u}$ is the solution of the problem. The essential simplification in dealing with the difference boundary value problem consists in that no difficulties arise in showing that the corresponding limit functions $\underline{u}$ and $\bar{u}$ "attain" the given boundary values; and, in view of the uniqueness of the solution of the problem, $\underline{u}$ must equal $\bar{u}$, so that the analogue of the second step automatically holds true, and it is only the first step which needs to be carried out.

The analogue of a theorem of Perron that we wish to prove is the following: the function $\bar{u}$, defined by

$$\bar{u}(m, n) = g \cdot l \cdot b \cdot [v(m, n)], \tag{4}$$

for (m, n) in $D + C$, where v is an arbitrary upper function for f, is the solution of the Dirichlet difference boundary value problem (1), (2). (Similarly, the function $\underline{u}$ defined by

$$\underline{u}(m, n) = l \cdot u \cdot b \cdot [w(m, n)], \tag{5}$$

for (m, n) in $D + C$, where w is an arbitrary lower function for f, is again the solution of the Dirichlet difference boundary value problem (1), (2)).

First of all, the definition (4) makes sense, and $\bar{u}$ is finite at each point of $D + C$, by virtue of the minimum principle for superharmonic functions. Also, $\bar{u}$ "takes on" the prescribed boundary values. If it were also known that $\bar{u}$ is superharmonic in $D + C$ then it would follow that $\bar{u}$ is harmonic in D. Because if

$$4\bar{u}(m, n) \geqq \bar{u}(m + 1, n) + \bar{u}(m - 1, n) + \bar{u}(m, n + 1) + \bar{u}(m, n - 1), \tag{6}$$

for (m, n) in D, and the equality sign did not hold in (6) for some point, call it (m, n), of D, then the function u_* defined by

$$u_*(P) = \begin{cases} \frac{1}{4}[\bar{u}(m+1, n) + \bar{u}(m-1, n) + \bar{u}(m, n+1) + \bar{u}(m, n-1)], \\ \bar{u}(P), \text{ for } P \text{ in } D + C, P \neq (m, n), \end{cases}$$

would also be an upper function for f; but, at (m, n)

$$\bar{u}_*(m, n) < \bar{u}(m, n),$$

contradicting the definition (4) of $\bar{u}$. Hence it only remains to prove (6), i.e. that $\bar{u}$ is superharmonic in D, and the argument will be complete.

Let $P_1, P_2, \cdots, P_d$ denote the d points of the interior D, and consider P_1 first. From the definition (4) of $\bar{u}$ it follows that there is a sequence of upper functions $v_1^{(1)}, v_2^{(1)}, v_3^{(1)} \cdots$ such that both

$$v_p^{(1)}(P_1) \geqq \bar{u}(P_1), p = 1, 2, 3, \cdots,$$

and

$$\lim_{p\to\infty} v_p^{(1)}(P_1) = \bar{u}(P_1).$$

Similarly, for each integer $i = 1, \cdots, d$, the definition (4) of $\bar{u}$ implies that there is a sequence of upper functions $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots$ such that both

$$v_p^{(i)}(P_i) \geqq \bar{u}(P_i), i = 1, \cdots, d; p = 1, 2, 3 \cdots, \tag{7}$$

$$\lim_{p\to\infty} v_p^{(i)}(P_i) = \bar{u}(P_i),$$

where the index i refers merely to the sequence of upper function which converges to $\bar{u}$ at the point P_i. Now introduce the sequence of functions $z_1, z_2, z_3, \cdots$ by defining, for each positive integer p, the function z_p by

$$z_p(P_i) = \min_{j=1,\cdots,d} v^{(j)}(P_i), i = 1, \cdots, d,$$

$$z_p(m, n) = f(m, n), \text{ for } (m, n) \text{ on } C,$$

so that, in effect, for $p = 1, 2, 3, \cdots$

$$z_p(m, n) = \min_{j=1,\cdots,d} v_p^{(j)}(m, n), \text{ for } (m, n) \text{ in } D + C,$$

or

$$z_p = \min \{v_p^{(1)}, \cdots, v_p^{(d)}\}.$$

For each $p = 1, 2, 3, \cdots$ and $j = 1, \cdots, d$, the function v_p is superharmonic, therefore it readily follows from (8) that z_p is also superharmonic; because, given $p = 1, 2, 3, \cdots$ and (m, n) in D, then, for some $q = 1, \cdots, d$:

$$\begin{aligned} 4z_p(m, n) &= 4v_p^{(q)}(m, n), \\ &\geqq v_p^{(q)}(m+1, n) + v_p^{(q)}(m-1, n) + v_p^{(q)}(m, n+1) + v_p^{(q)}(m, n-1), \\ &\geqq z_p(m+1, n) + z_p(m-1, n) + z_p(m, n+1) + z_p(m, n-1). \end{aligned} \tag{9}$$

Besides, from (7) and (8) it is seen that

$$\lim_{p\to\infty} z_p(m, n) = \bar{u}(m, n),$$

for (m, n) in $D + C$. Consequently, the function $\bar{u}$ satisfies (6). A similar argument holds for $\underline{u}$.

The preceding argument shows directly, independently of any considerations involving Cramer's rule for linear equations, that $\bar{u}$ is the solution of the Dirichlet problem. However, if, as in the introduction, it is assumed that the existence of the solution u of the Dirichlet problem has already been demonstrated, then the fact that $\bar{u}$ is the solution follows easily, since the solution u is itself an upper function for f. Similarly for $\underline{u}$.

In conclusion, it is pointed out that the considerations of this and the preceding note [2] carry over immediately to difference equations of the form

$$u(m, n) = a(m, n)\, u(m + 1, n) + b(m, n)\, u(m - 1, n) + c(m, n)\, u(m, n + 1) + d(m, n)\, u(m, n - 1),$$

where

$$a(m, n), b(m, n), c(m, n), d(m, n) > 0,$$

and

$$a(m, n) + b(m, n) + c(m, n) + d(m, n) = 1,$$

for each point (m, n) of D. Difference equations of this type arise in probability questions (see, for example, J. H. Curtiss [1]).

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AN ESTIMATE OF VISCOSITY EFFECT ON THE HYPERSONIC FLOW OVER AN INSULATED WEDGE*

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1. Introduction. For hypersonic flow over a slender wedge, the usual neglect of viscosity leads to the result that the shock wave is a straight one lying very close to the surface of the body. The ratio of the shock wave angle to the vertex angle of the body depends only on the hypersonic parameter $K = M^0\beta$, where M^0 is the free stream Mach number and β is the semi-vertex angle of the body. As the value of β is assumed to be small by the definition of hypersonic flow, the absolute thickness of the fluid between the shock wave and the body surface is therefore very small. Evidently the effect of a viscous boundary layer would be felt. The present paper is concerned with an estimation of the viscosity effect in a particularly simple problem, namely, the insulated wedge with the Prandtl number of air assumed to be unity, by means of the "Pohlhausen method" well known in the solution of ordinary boundary layer problems. The flow is assumed to be laminar, and the behavior of air is assumed to follow the same laws as under ordinary conditions.

The results indicate that the Reynolds number of the flow enters as a scale factor to the length only. As a consequence at different Reynolds numbers the shock shapes are similar, provided the hypersonic parameter K remains unchanged. On a given body, the absolute length over which the viscosity effect is important increases, as expected, when the Reynolds number is low and the hypersonic parameter high. For example, it is found that for $K = 1$, $\beta = 5^0$, in actual flight in free air the viscosity effect might still be negligible. But in the extreme case of a hypothetical hypersonic wind tunnel which obtains the desired Mach number by expanding the air from nearly atmospheric conditions, the Reynolds number may become so low that a discernible curved shock occurs. In such cases, the drag measurements, excluding the contribution due to base drag, would be very much different from the pressure drag of inviscid theory.† The present analysis, though crude, perhaps could help to correlate between wind tunnel and free air results, or at least give a better interpretation.

2. Formulation of the Problem. In estimating the viscosity effect on the shock wave in hypersonic flow over a thin wedge, one may first recall the solution of the viscous boundary layer over such bodies when the speed is in the ordinary supersonic range. There, as the shock wave is at a large distance from the body surface in comparison with the thickness of the viscous boundary layer, the mutual interaction is unimportant. In the present case, the interaction can no longer be neglected since the thickness between the shock wave and the body is of the same order of magnitude as the ordinary "viscous" boundary layer, at least over a certain length along the body.

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† Details of these are given in ref. 1, and omitted in this paper.

The type of flow between the shock wave and the body may also be described by the same simplified hydrodynamic equations usually adopted for boundary layer problems. The order of magnitude arguments which lead to the simplifications may be carried through in a similar manner.[2] The only difference from the ordinary boundary layer problems lies in the boundary conditions at the outer edge of the layer. The flow variables there must now be connected to the free stream values by means of the shock relations, suitably modified for hypersonic flow. They consequently depend on the unknown shape of the outer edge of the layer.

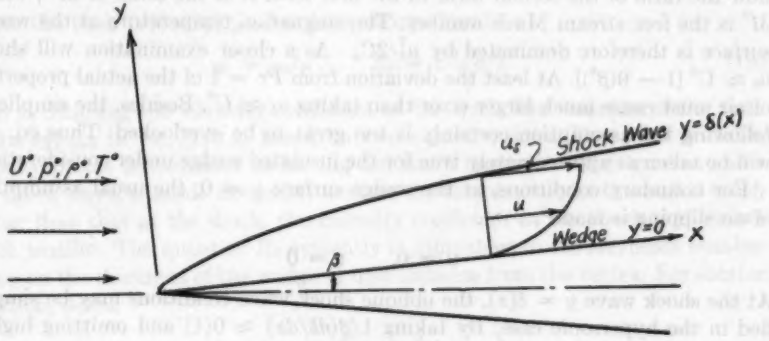

FIG. 1. General notations

Consider now the flow within the thin layer between the body surface and the shock wave above it (Fig. 1). The usual simplified differential equations for boundary layer are as follows:

$$\rho[u(\partial/\partial x) + v(\partial/\partial y)]u = -\partial p/\partial x + (\partial/\partial y)\,(\mu\partial u/\partial y) \tag{1}$$

$$\partial p/\partial y = 0 \tag{2}$$

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y = 0 \tag{3}$$

$$\rho[u(\partial/\partial x) + v(\partial/\partial y)]\,(CpT) = u(\partial p/\partial x) + (\partial/\partial y)\,(\lambda\partial T/\partial y) + \mu(\partial u/\partial y)^2 \tag{4}$$

In eqs. (1) to (4), p, ρ and T are the pressure, density and temperature of the fluid; u and v are the x- and y-components of the velocity; μ is the viscosity coefficient, λ the heat conduction coefficient, and C_p the specific heat at constant pressure. The x-axis is, for convenience, taken along the surface of the wedge, and the y-axis perpendicular to it directing toward the fluid. The free stream velocity U^0 is now at an angle $-\beta$ with respect to the x-axis, β being the semi-vertex angle of the wedge. Among the variables p, ρ and T the perfect gas law is assumed to hold

$$p = \rho RT, \tag{5}$$

R being the gas constant.

It is well known that if the Prandtl number Pr equals unity and C_p is constant, there exists a particular integral

$$T + (u^2/2C_p) = \text{const.}, \tag{6}$$

which corresponds to an insulated body surface. In the case of an insulated wedge in hypersonic flow, (6) would require at the shock wave

$$T_\delta + (u_\delta^2/2C_p) = \text{const.} = T_0 \tag{7}$$

where T_0 equals the stagnation temperature occurring at the wedge surface, and quantities with subscript "δ" denote those at the shock wave. Since both T_δ and u_δ are functions of the shock wave slope $d\delta/dx$ through the oblique shock conditions, the satisfaction of eq. (7) strictly speaking is possible only for a unique relation between $d\delta/dx$ and T_0. However, one sees that for hypersonic flow the ratio of the second term to the first term is of the order of M^{0^2}, where M^0 is the free stream Mach number. The stagnation temperature at the wedge surface is therefore dominated by $u_\delta^2/2C_p$. As a closer examination will show, $u_\delta \approx U^0\,[1 - 0(\beta^2)]$. At least the deviation from $Pr = 1$ of the actual properties of air must cause much larger error than taking $u_\delta \approx U^0$. Besides, the simplicity following the assumption certainly is too great to be overlooked. Thus eq. (6) will be taken as approximately true for the insulated wedge under consideration.

For boundary conditions, at the wedge surface $y = 0$, the usual assumption of no slipping is made*,

$$u = 0, \qquad v = 0 \tag{8}$$

At the shock wave $y = \delta(x)$, the oblique shock wave conditions may be simplified in the hypersonic case. By taking $1/\beta(d\delta/dx) \approx 0(1)$ and omitting higher order terms, they reduce to the following:

$$p_\delta^* = p_\delta/p^0 = \frac{2\gamma K^2}{\gamma+1}\left(1 + \frac{1}{\beta}\frac{d\delta}{dx}\right)^2 - \frac{\gamma-1}{\gamma+1} \tag{9}$$

$$\frac{1}{\rho_\delta^*} = \rho^0/\rho_\delta = \frac{\gamma-1}{\gamma+1} - \frac{2}{(\gamma+1)K^2[1 + (d\delta/dx)/\beta]^2} \tag{10}$$

$$u_\delta^* = u_\delta/U^0 = 1 - \frac{\beta^2}{2}\left[1 - \frac{4}{\gamma-1}\frac{1}{\beta}\frac{d\delta}{dx}\frac{1 - K^2[1 + (d\delta/dx)/\beta]^2}{K^2[1 + (d\delta/dx)/\beta]^2}\right] \tag{11}$$

$$v_\delta^* = v_\delta/U^0 = \beta\left[\frac{2}{\gamma+1}\frac{K^2[1 + (d\delta/dx)/\beta]^2 - 1}{K^2[1 + (d\delta/dx)/\beta]^2} - 1\right] \tag{12}$$

where the superscript "0" denotes the free stream value, and $K = M^0\beta$ is the hypersonic parameter. The expression of $T_\delta^* = T_\delta/T^0 = p_\delta^*/\rho_\delta^*$ may be easily derived from above. From the complicated form of eqs. (9) to (12), it is clear that unless numerical procedure is resorted to, the solution is very difficult.

For an approximate solution, the integrated form of eq. (1) may be used. Integrating with respect to y from 0 to δ and taking into account eqs. (2) and (3), one obtains

$$(\partial/\partial x)\left[\int_0^\delta \rho u(u - u_\delta)\,dy\right] = -(dp/dx)\delta + (\mu(\partial u/\partial y))_{y=\delta} - (\mu(\partial u/\partial y))_{y=0} \tag{13}$$

* Effect of slipping is the same as a different velocity profile, and would in general be small as shown in section III.

Except for the second term on the right, eq. (13) is exactly the same as for ordinary boundary layer problems. It is still the momentum integral of the flow. Introduce next the following transformations:

$$x^* = x\beta, \qquad y^* = y/\delta, \qquad \rho^* = \rho/\rho^0 \tag{14}$$
$$u^* = u/U^0, \qquad p^* = p/p^0, \qquad \mu^* = \mu/\mu^0$$

The equation becomes

$$\frac{\partial}{\partial x^*}\left[\delta\rho_s^* u_s^{*2}\int_0^1 \frac{\rho}{\rho_s}\frac{u}{u_s}\left(\frac{u}{u_s}-1\right)dy^*\right] = -\frac{\delta}{\gamma M^{02}}\frac{dp^*}{dx^*} - \frac{\mu_0^*}{\delta R_\beta}\left[\frac{\partial}{\partial y}\left(\frac{u}{u_s}\right)\right]_{y=0} \tag{15}$$

where the term $(\mu(\partial u/\partial y))_{y=\delta}$ has been dropped and

$$\mu_0^* = \mu_0/\mu^0, \qquad R_\beta = U^0\rho^0\beta/\mu^0 \tag{16}$$

with μ_0 denoting the viscosity coefficient at the wall. The omission of the term $(u(\partial u/\partial y))_{y=\delta}$ in eq. (16) is based on two considerations. First, the probable velocity profile is visualized as having but small normal velocity gradient near the shock wave, if at all. Secondly, because the temperature at the wall is much higher than that at the shock, the viscosity coefficient at the shock is therefore much smaller. The quantity R_β evidently is equivalent to the Reynolds number based on the thickness of the wedge at unit distance from the vertex. For shorter notation, define

$$m = -\rho_s^* u_s^{*2}\int_0^1 \frac{\rho}{\rho_s}\frac{u}{u_s}\left(\frac{u}{u_s}-1\right)dy^* \tag{17}$$

Eq. (15) may be rewritten as

$$\frac{d}{dx^*}(m\delta) = \frac{\delta}{\gamma M^{02}}\frac{dp^*}{dx^*} + \frac{\mu_0^*}{\delta R_\beta}\left[\frac{\partial}{\partial y}\left(\frac{u}{u_s}\right)\right]_{y=0} \tag{15}'$$

The quantity $m\delta$ is the usual "momentum thickness".

To assume an appropriate velocity profile for the evaluation of the integral involved in eq. (17), the task is still in general rather difficult. Qualitatively one may argue that the fluid between the shock and the body could be regarded as consisting of a viscous sublayer of approximately the same nature as in the ordinary boundary layer. The growth of this sublayer tends to be rather rapid at first, but becomes very slow far away from the vertex. Therefore at distances large enough, the inviscid solution of a straight shock wave must be asymptotically correct. The true velocity profile across the entire thickness between the shock and the body must approach the inviscid one of constant velocity. However, in the range where the interaction between shock wave and viscous effects are important, the asympototic behavior need not be too much stressed.* Here the viscous boundary layer may be regarded as extending throughout the thickness from the body to the shock wave. The present problem then is just

* The asymptotic behavior of the boundary layer over a flat plate in hypersonic flow has been treated by Lees and Probstein.[3]

like an ordinary boundary layer; the main feature is only a varying pressure gradient depending on the growth. As a simplest procedure, one may adopt a linear profile at all stations along the surface. Under no pressure gradient, the linear profile was found to be a good approximation for ordinary boundary layer at high Mach number.[6]

Let then

$$u/u_s = y^* \tag{18}$$

It follows from eq. (6),

$$\rho/\rho_s = T_s/T \approx [2/(\gamma - 1)]\,(T_s^*/M^{02})\,(1 - y^{*2})^{-1} \tag{19}$$

and, with a little calculation,

$$m = m_1\, p^* u_s^{*2}/M^{02} \text{ where } m_1 = [2/(\gamma - 1)]\,(1 - \ln 2) \tag{20}$$

Since $u_s^* \approx 1$, eq. (15)′ becomes

$$m_1\, d(p^*\delta)/dx^* = (\delta/\gamma)(dp^*/dx^*) + (M^{02}\mu_0^*/R_\delta)\delta^{-1} \tag{21}$$

Introduce now the length scale

$$L = M^{02}\mu_0^*/R_\delta \tag{22}$$

and the non-dimensionalizing transformations

$$\xi = x^*/L,\dagger \qquad \eta = \delta/L \tag{23}$$

Eq. (21) may then be formulated into an integro-differential equation:

$$\eta^2 = (2/m_1)(p^*)^{-\alpha} \int_0^\xi (p^*)^{\alpha-1}\, d\xi \tag{24}$$

where $\alpha = 2[1 - (1/\gamma m_1)]$ and, by eq. (9)

$$p^* = [2\gamma K^2/(\gamma + 1)]\,\{(d\eta/d\xi)^2 + 2(d\eta/d\xi) + 1 - [(\gamma - 1)/2\gamma K^2]\} \tag{25}$$

In eq. (24), the condition of attached shock has been used, i.e., $\eta = 0$ at $\xi = 0$. Deviations from this idealized condition in a small region near the tip will not affect the results at the distances where the boundary layer equations hold, just like in the ordinary boundary layer problem.

The solution is seen to involve only the hypersonic parameter K. Different wedges and free stream conditions affect the result only through the length scale L.

On the other hand, the above formulation of the problem does not give the correct asymptotic behavior of approaching the inviscid solution far away. This is inherent because of the assumption of a linear velocity profile. A brief discussion of the asymptotic behavior of eq. (21) is given in the following section.

3. Method of Solution and Results. In solving eq. (24). it is noted that the value of α is 1.07 if γ is taken to be 1.405. The function behind the integral

† It may be noted that $\xi \equiv \beta^3 R_x/M^{02}\mu_0^*$, if R_x is the Reynolds number based on distance x and free stream conditions.

sign in eq. (24) thus differs but slightly from unity over practically the entire range except very close to the origin. A natural way to proceed is to regard $\alpha - 1$ as a small parameter and to expand the solution in ascending powers of $\alpha - 1$. The zeroth order equation thus obtained is,

$$\eta_0^2[(d\eta_0/d\xi)^2 + 2(\partial\eta_0/\partial\xi) + A] = B\xi \tag{26}$$

where $A = 1 - [(\gamma - 1)/2\gamma K^2]$, $B = (\gamma + 1)/m_1\gamma K^2$. The higher order equations contain the solutions of the lower order equations. Eq. (26) will now be examined in detail. The correction in terms of $(\alpha - 1)$ will be discussed later.

One observes that, besides being non-linear, (26) also has a singularity at the origin from which the solution is to start. Since the desired solution should exhibit large slopes near the origin, for $(\xi, \eta_0) \to 0$, there must be $(d\eta_0/d\xi) \approx B(\xi/\eta_0^2)$, or

$$\eta_0 \approx (2/\sqrt{3})\, B^{1/4}\xi^{3/4} \tag{27}$$

A change of variable may now be made: Let

$$\zeta = \xi^{1/4} \tag{28}$$

Then (26) is formally satisfied by a power series in ζ:

$$\eta_0 = \zeta^3 \sum_{n=0}^{\infty} a_n\zeta^n \tag{29}$$

where the a_n's are found to be the following:

$$\begin{aligned}
a_0 &= (2/\sqrt{3})B^{1/4},\ a_1 = -\tfrac{4}{7} \\
a_2 &= -\tfrac{1}{3}a_0^{-1}[\tfrac{12}{10}a_1^2 + 5a_1 + A] \\
a_3 &= -\tfrac{8}{27}a_0^{-2}[2a_1A + \tfrac{11}{2}(a_1^2 + a_0a_2) + \tfrac{23}{6}a_0a_1a_2 + \tfrac{7}{12}a_1^3] \\
a_4 &= -\tfrac{4}{135}a_0^{-3}[(\tfrac{1358}{180}a_1 + 6)a_0^2a_2 + (\tfrac{94}{18}a_0a_2 + 12a_1 + 2A + \tfrac{1490}{180}a_1^2)a_0a_2 + \\
&\qquad a_1^2(a_1^2 + 2a_1 + A)]
\end{aligned} \tag{30}$$

etc.

The correction to the zeroth order solution may be estimated as follows. Integrating eq. (24) by parts, one has

$$\tfrac{1}{2}m_1\eta^2 = (\xi/p^*) - (\alpha - 1)(p^*)^{-\alpha}\int_0^{\xi}(p^*)^{\alpha-2}(dp^*/d\xi)\xi\, d\xi \tag{31}$$

The zeroth order solution actually corresponds to putting $\alpha - 1 = 0$ in (31). Thus when $\alpha - 1 \ll 1$, the second term in the right-hand side may be estimated by means of the zeroth order solution. With $\epsilon = \alpha - 1$ and subscript "0" indicating the zeroth order solution, it follows

$$\eta_1^2 \approx \eta_0^2\left\{1 - (\epsilon/\xi)\int_0^{\xi}[1 - 2(\xi/\eta_0)(d\eta_0/d\xi)]\, d\xi + 0(\epsilon^2)\right\} \tag{32}$$

provided $\epsilon\,|\ln p^*| \ll 1$. The last restriction is readily seen to be true for all points where the fundamental boundary layer concept is expected to hold. Expanding eq. (32) by using eq. (29), one gets

$$\eta_1 \approx \eta_0(1 + k\,\epsilon) \tag{33}$$

where

$$k = \tfrac{1}{2}(1 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + \cdots)$$

and

$$\begin{aligned} b_1 &= \tfrac{1}{2}(a_1/a_0) \\ b_2 &= \tfrac{1}{3}[2(a_2/a_0) - (a_1/a_0)^2] \\ b_3 &= \tfrac{1}{4}[3(a_3/a_0) - 3(a_1a_2/a_0^2) - (a_1/a_0)^3] \end{aligned} \tag{34}$$

etc.

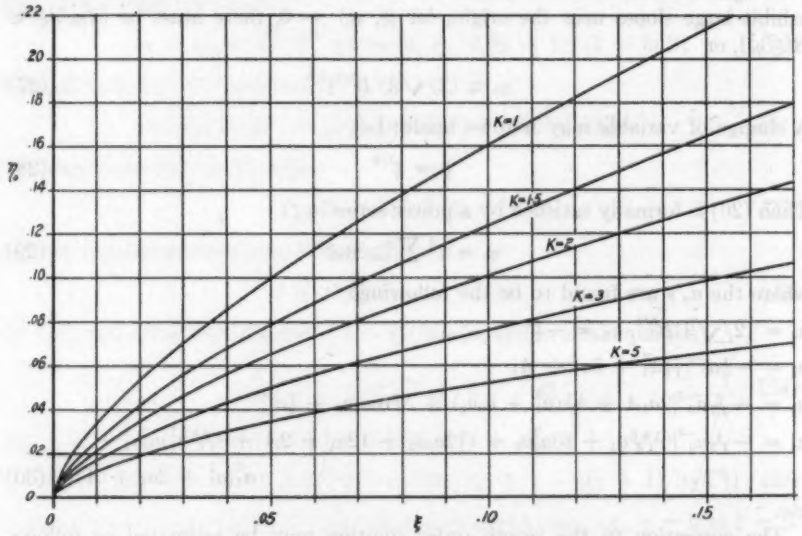

FIG. 2. Series solutions for η_0 at various K's

It may be pointed out here that for consistency one should stop at $b_m\zeta^m$ when $b_m\zeta^m \approx 0(\epsilon)$. Also the values of η_0 should only be computed to an accuracy of the order of ϵ^2, i.e., .005 for $\epsilon = .07$.

Actual computations of the zeroth order solution for the cases of $K = 1$, 1.5, 2, 3 and 5 have been carried out by using eq. (30) and taking terms up to $a_4\zeta^4$ in the series. As K increases, the practical convergence becomes poor for moderate arguments of ζ, say $\zeta \geqq 0.5$. The number of terms taken is found to be insufficient to yield accurate results for the cases $K \geqq 2$ to cover the whole range where viscosity effect is present (see below). The results, nevertheless, are plotted as Fig. 2. The corrections k in these cases are found to be of the order of 1% for the range of interest. The approximate nature of the method certainly would justify the omission of the corrections. In the following only the zeroth order solution will be discussed.

For the behavior as $K \to \infty$*, make the transformation

$$\zeta_1 = \sqrt{K}\zeta \tag{35}$$

Then eq. (29) takes the form

$$\eta_0 K^2 = a'\zeta_1^2 \sum_{n=0}^{\infty} [a'_n/(a')^n]\, \zeta_1^n \text{ where } a'_n = a_n a_0^{n-1} \text{ and } a' = a_0\sqrt{K}. \tag{36}$$

Thus,

$$d\eta_0/d\xi = (d\eta_0/d\zeta_1)(d\zeta_1/d\xi) = (a'/4\zeta_1) \sum_{n=0}^{\infty} (n+3)\, [a/(a'_1)^n]\zeta_1^n \tag{37}$$

When $K \to \infty$, $A \to 1$, and the coefficients in eq. (37) become independent of the value of K. Actual computation, however, indicates that the convergence of both eqs. (36) and (37) are slow when ζ_1 becomes comparable with unity.

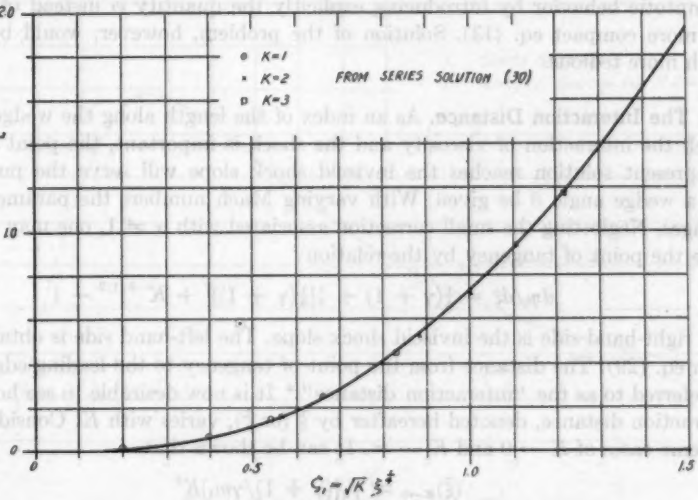

FIG 3 SOLUTION OF THE ZEROTH ORDER EQUATION FOR $K \to \infty$

Instead, the method of numerical integration is used. With $\vartheta \equiv \eta_0 K^2$, the results are plotted as Fig. 3 in the form ϑ vs. $K^2\xi$. When one plots the results for $K = 1$, 1.5, etc. in the variables ϑ and $K^2\xi$, even for $K = 1$ the difference from the case of $K \to \infty$ was found to be only in the third decimal place. It is therefore obvious that for cases of $K \geqq 2$ the solution is well approximated by the solution for $K \to \infty$. Indeed, due to the slow convergence of the series (29) for larger values of K, the step-by-step solution for $K \to \infty$ is probably more accurate in the cases of $K = 3$ and 5 then those in Fig. 2.

The nature of the solution at very large distances may also be discussed on the basis of the zeroth order equation. If one stipulates that the slope approaches

* The solution is good even for $K \gg 1$ (see ref. 2).

a finite value at infinity, it follows that asymptotically $\eta_0^2 \propto \xi$ which yields, however,

$$d\eta_0/d\xi \propto 1/\sqrt{\xi}$$

Thus the asymptotic behavior does not correspond to the inviscid value as argued from a physical point of view. The present solution would hold up to approximately the point where the slope of the shock reaches the inviscid value, i.e., when it merges into the correct asymptotic solution.

It may be remarked that the above incorrect asymptotic behavior stems from the emphasis on the momentum relation along the surface. The normal velocity component being the only factor in the determination of the inviscid shock slope, its suppression in eq. (13) cannot but lead to an erroneous result unless the velocity profile is *exact*. Thus one may expect an improvement of the asymptotic behavior by introducing explicitly the quantity v_s instead of using the more compact eq. (13). Solution of the problem, however, would become much more tedious.

4. The Interaction Distance. As an index of the length along the wedge over which the interaction of viscosity and the shock is important, the point where the present solution reaches the inviscid shock slope will serve the purpose. Let a wedge angle β be given. With varying Mach numbers the parameter K changes. Neglecting the small correction associated with $\alpha \neq 1$, one may determine the point of tangency by the relation

$$d\eta_0/d\xi = \tfrac{1}{2}(\gamma + 1) + \{[\tfrac{1}{2}(\gamma + 1)]^2 + K^{-2}\}^{1/2} - 1 \tag{38}$$

The right-hand side is the inviscid shock slope. The left-hand side is obtainable from eq. (29). The distance from the point of tangency to the leading edge will be referred to as the "interaction distance".* It is now desirable to see how the interaction distance, denoted hereafter by $\bar{\xi}$ (or $\bar{\zeta}$), varies with K. Consider the extreme cases of $K \to 0$ and $K \to \infty$. It can be shown that

$$(\bar{\xi})_{K\to 0} \approx \tfrac{9}{16}[(\gamma + 1)/\gamma m_1]K^2 \tag{39}$$

and, for the case of $K \to \infty$, interpolating from the numerical data one gets $\bar{\zeta}_1 = 1.589$ or

$$(\bar{\xi})_{K\to\infty} \approx 6.38/K^2 \tag{40}$$

For moderate values of K, $\bar{\xi}$ must be computed. The results of $\bar{\xi}$ vs. K are plotted in Fig. 4. Physically, if L is fixed, an increase of K is followed by a flatter shock wave, though excluding the cases of $K \leqq 1$ again a universal curve (Fig. 3) prevails. The length of the interaction distance increases with K. In fact, it can be readily shown that taking

$$\mu_0^* = (T_0^*)^{\epsilon} \approx [\tfrac{1}{2}(\gamma - 1)M^{0^2}]^{\epsilon}$$

* The significance of such a distance is much the same as the ordinary boundary layer "thickness," expressing the extent of the viscous effect.

where $s = 0.76$ for air in ordinary temperature range or $s = 0.50$ from kinetic theory for simple spherical molecules, one obtains

$$\beta^{4+2s}R_s \approx [\tfrac{1}{2}(\gamma - 1)]^s K^{2+2s}\bar{\xi} \tag{41}$$

where $R_s = \rho^0 U^0 \bar{x}/\mu^0$, i.e., *the Reynolds number of the point of tangency based on free stream values of velocity and kinematic viscosity*. Eq. (41) has been plotted for both $s = 0.76$ and $s = \frac{1}{2}$ in Fig. 5.

It is quite clear that in comparison between hypersonic flows observed in the wind tunnel and in flight, one must keep in mind the "scale effect" represented by the scale L. Unless L is very small as measured against the total

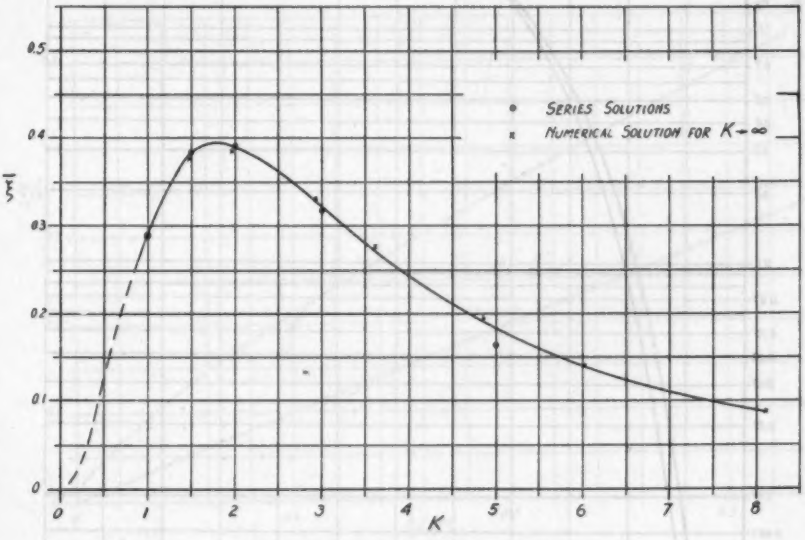

FIG. 4. Interaction distance vs. K

length of the wedge, the phenomena could be quite different. Examples of such application of the theory were given in ref. 1.

5. The Special Case of a Flat Plate. An interesting special case is the flat plate, i.e., $\beta \equiv 0$. Because now $A \propto 1/K^2$ and approaching infinity, difficulty is encountered in trying to derive the answer from previous results. However, if one goes back to eqs. (9) and (21) and takes out the factor β in both equations, there follows respectively

$$p^* = [2\gamma/(\gamma + 1)]M^{0^2}\{(d\delta/dx)^2 - [(\gamma - 1)/2\gamma M^{0^2}]\} \tag{42}$$

$$\delta(d\delta/dx) + \delta^2[1 - (1/\gamma m_1)]d \ln p^*/dx = M^{0^2}\mu_0^*/R_1 m_1 p^* \tag{43}$$

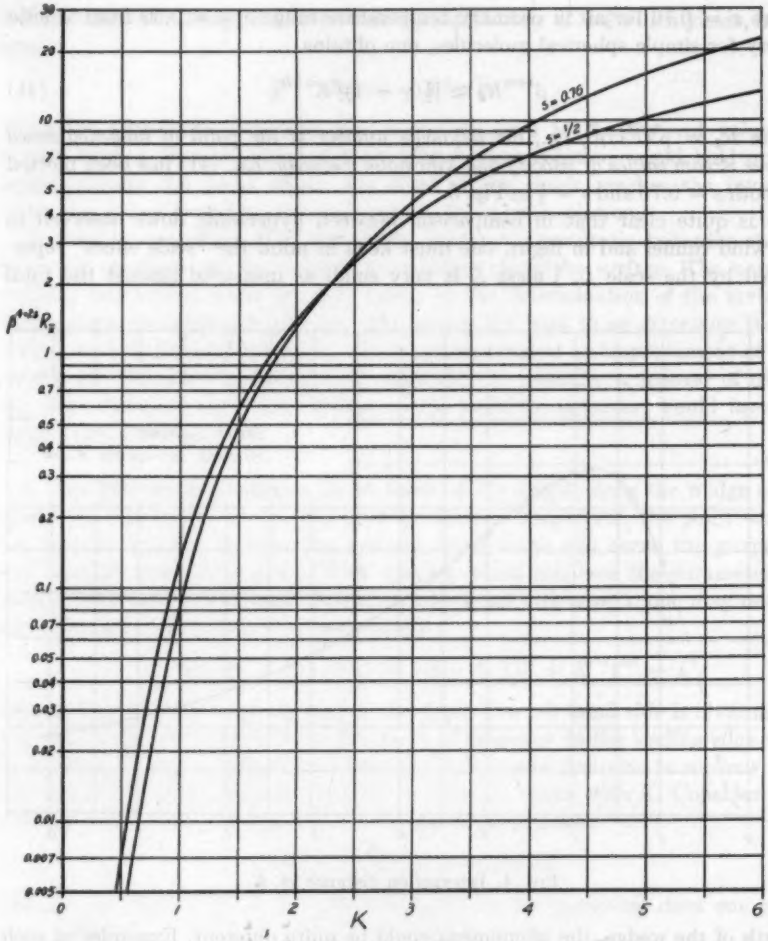

FIG. 5. $\beta^{s+2s}R_\xi$ vs K

where $R_1 = U^0\rho^0/\mu^0$, i.e., the Reynolds number based on a length of unity. In parallel with eqs. (22) and (23), let the length scale be

$$L_1 = M^{0^2}\mu_0^*/R_1 \tag{44}$$

and

$$\xi = x/L_1,^* \qquad \eta = \delta/L_1 \tag{45}$$

then an equation exactly the same as eq. (24) results. The zeroth order approximation is now given by

$$d\eta_0/d\xi = [(\gamma + 1)/\gamma m_1 M^{0^2}](\xi/\eta_0^2) + [\gamma - 1)/2\gamma M^{0^2}] \tag{46}$$

* By definition (44), $\xi \equiv R_x/M^{0^2}\mu_0^*$, R_x being the Reynolds number based on distance x and free stream conditions.

The solution of eq. (46) evidently is of the same nature as that of eq. (26). With $\zeta = \xi^{1/4}/\sqrt{M^0}$, there follows

$$\eta_0/M^0 = \zeta^3 \sum_{n=0}^{\infty} c_n \zeta^n \tag{47}$$

where the c_n's are given by

$$\begin{aligned} c_0 &= (2/\sqrt{3})B_1^{1/4}, \\ c_1 &= 0, \\ c_2 &= -A_1/3c_0, \\ c_3 &= 0 \\ c_4 &= -\tfrac{4}{15}(c_2/c_0^2)(\tfrac{47}{9}c_0c_2 - 2A_1) \end{aligned} \tag{48}$$

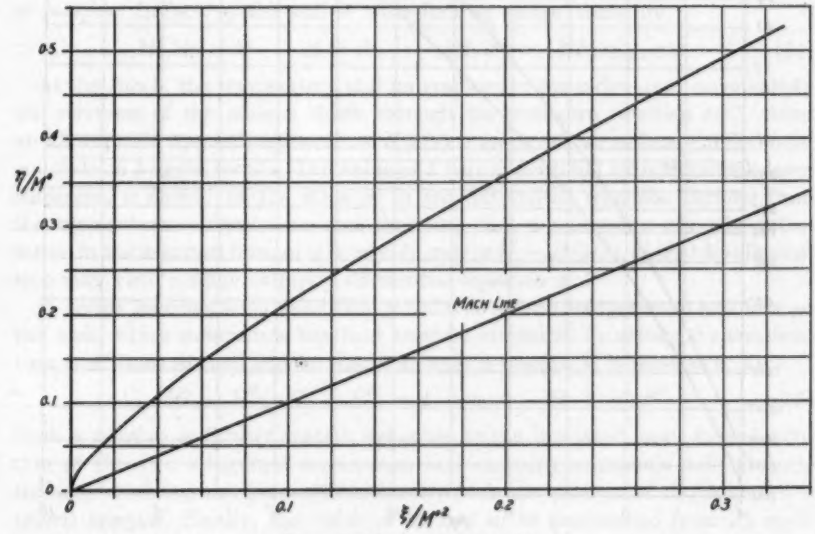

FIG. 6. Solution for flat plate

where $A_1 = -[(\gamma - 1)/2\gamma]$, $B_1 = [(\gamma + 1)/\gamma m_1]$

The correction for $\alpha \neq 1$ formally takes the same form as eqs. (33) and (34), except that the meanings of ξ and η are changed. The computed results of the flat plate based on eq. (47) are presented as Fig. 6. The solution is assumed to hold until the slope reaches approximately the Mach angle. To find the values $\bar{\xi}$ at the tangent point, it is noted that explicitly eqs. (47) and (48) yield

$$\eta_0/M^0 = \zeta_1^3\,(1.1904 + .04043\,\zeta_1^3 - .00005\,\zeta_1^4 + \cdots) \tag{49}$$

Evidently the first two terms would already be a good approximation. The condition of Mach angle being

$$d\eta_0/d\xi = 1/(M^{0^2} - 1)^{1/2} \approx 1/M^0$$

in hypersonic case, the value $\bar{\zeta}_1$ may be solved from

$$[d(\eta_0/M_0)/d\zeta_1]_{\zeta_1=\bar{\zeta}_1} = 4\bar{\zeta}_1^3 \tag{50}$$

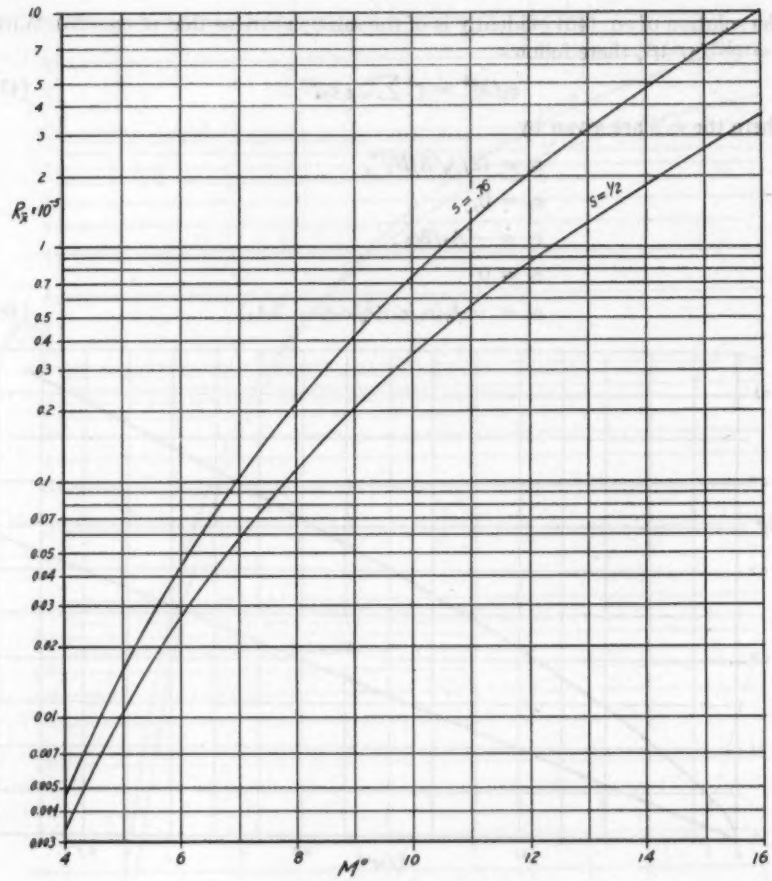

FIG. 7. $R_{\bar{x}}$ vs. M^0 for flat plate

Differentiating eq. (49) and keeping only the first two terms, one substitutes the result into the left-hand side of eq. (50). It then follows that $\bar{\xi}_1 \approx .936$ i.e.,

$$\bar{\xi} \approx .768\, M^{02} \tag{51}$$

Hence the interaction distance increases much more rapidly with M^0 than the corresponding variable in the case of a wedge.

With the assumption of hypersonic Mach numbers, the physical length is given by

$$R_{\bar{x}} \approx .768\, [\tfrac{1}{2}(\gamma - 1)]^{\epsilon} (M^0)^{4+2\epsilon} \tag{52}$$

Eq. (52) is plotted as Fig. 7.

6. Extension to a Heat-conducting Surface and $Pr \neq 1$. The considerations of $Pr \neq 1$ and a heat conducting wedge do not lead to fundamental difficulties. The

energy eq. (4) involving the enthalpy now cannot be omitted, as eq. (6) ceases to be a particular solution. But a similar technique may still be applied in using an integrated form across the thickness of the shock layer. Integrating eq. (4), one finds readily

$$(\partial/\partial x)\int_0^\delta \rho u(H - H_s)\,dy = (dp/dx)\int_0^\delta u\,dy - (\lambda/C_p)(\partial H/\partial y)\Big|_0^\delta + \int_0^\delta \mu(\partial u/\partial y)^2\,dy \quad (53)$$

where $H = C_pT$ = enthalpy, and C_p is taken to be constant. This equation is to be used with eq. (13) simultaneously. The solution now involves the representation of H as a polynomial satisfying as many of the boundary conditions as possible. In fact, at the wall of a conducting wedge, there are

$$\partial H/\partial x = 0, \qquad H = H_0, \qquad \partial H/\partial y = (\partial H/\partial y)_0, \text{ etc.} \quad (54)$$

At the shock, the temperature and its gradient in the x-direction must satisfy the relations of the oblique shock through the unknown function $\delta(x)$. After an appropriate approximation $H = H(y/\delta)$ is assumed, an ordinary differential equation in δ again results. The value of δ being identified with the shock layer thickness, it should be the same as in the differential equation derived from the momentum integral. One sees therefore that a parameter must be introduced in the assumed form of $u = u(y/\delta)$ and/or $H = H(y/\delta)$, so that its elimination may yield a single ordinary differential equation in δ.

It seems reasonable to take this parameter as the temperature gradient at the wall, which determines the heat transfer situation. To satisfy the temperature conditions at wall and at shock, at least a quadratic is necessary, e.g.,

$$(H - H^0)/(H_0 - H^0) = 1 + d_1y^* - (1 + d_1)y^{*2} \quad (55)$$

Such a relation is further readily reducible to the insulated case. Actual solution of the two integrated momentum and enthalpy equations will be quite tedious, involving integral expressions in which the parameter d_1 (unknown a priori) appears. Finally, the value of H_0 has to be determined from an equilibrium consideration, such as equating the heat input to the wedge through conduction and the loss through radiation.

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BOUNDARY LAYER CONTROL BY CONTINUOUS SURFACE SUCTION OR INJECTION

By T. Paul Torda

Introduction. The increase in maximum and cruising speeds has adversely affected the low-speed performance of modern aircraft. Research workers have been interested for a long time in the possibility of improving the take-off and landing characteristics of airplanes by boundary layer control. As early as 1929, J. Ackeret carried out experiments with suction wings, and since then both in this country and in Europe several workers have investigated boundary layer control by continuous surface suction or by suction through slots. Continuous surface suction is the more efficient of the two methods. All previous investigations were restricted to solutions for a given suction velocity distribution (either constant or linearly varying along the chord). Theodore von Kármán proposed the present investigation of the general case where the suction velocity distribution is not prescribed *a priori* but is calculated from the required behavior of the boundary layer.

Pohlhausen applied the von Kármán integral principle [1] to the evaluation of laminar boundary layers [2]. In the present paper, Pohlhausen's method is modified to apply to the general case of boundary layer control by continuous surface suction. First, this modified method is presented and its shortcomings pointed out. Then, a different method is developed using more appropriate boundary conditions. This new procedure eliminates all limitations of the modified Pohlhausen method. An illustrative example is included at the end of the paper.

It is thought that the general solution of boundary layer control by continuous surface suction is presented for the first time. The method applies to sweat-cooling by fluid injection without modification or restrictions.

The modified Pohlhausen method. In the analysis of the general case of boundary layer control by continuous surface suction the aim is to calculate the necessary suction velocity distribution for an appropriately prescribed boundary layer thickness variation, or, conversely, if a certain suction velocity distribution is desired, it should be possible to obtain the resulting boundary layer thickness variation.

The Pohlhausen method is well-known and, therefore, only the basic equations and the main steps in the modified analysis are given here. However, the limitations of the analysis are discussed in detail.

The boundary layer momentum equations is

$$u(\partial u/\partial s) + v(\partial u/\partial n) = -(1/\rho)(\partial p/\partial s) + \nu(\partial^2 u/\partial n^2) \tag{1}$$

The continuity equation is

$$\partial u/\partial s + \partial v/\partial n = 0 \tag{2}$$

The Bernoulli equation for the potential flow yields

$$-(1/\rho)(\partial p/\partial s) = U(dU/ds) \tag{3}$$

There will be some slip along the surface. This makes the introduction of a modifying function for the shear stress necessary. Thus,

$$(\tau)_{\text{porous wall}} = \epsilon(s)(\tau)_{\text{solid wall}} \tag{4}$$

In the above equations, s, n are rectangular coordinates along and perpendicular to the surface, respectively, u, v are the velocity components in the boundary layer in the directions of s and n, respectively, ρ is the density, p is the pressure, ν is the kinematic viscosity, U is the velocity in the potential flow parallel to the surface, τ is the shearing stress, and ϵ is a modifying function.

Integration of eq. (1) from the porous wall, $n = 0$, to the edge of the boundary layer, $n = \delta$, yields the extension of the von Kármán integral equation for the cases of fluid suction or injection through the wall. This equation is

$$(d/ds)\int_0^\delta u^2\,dn - U(d/ds)\int_0^\delta u\,dn = \delta U(dU/ds) - Uv_0 - \epsilon\nu(\partial u/\partial n)_0 \tag{5}$$

Here, and in the following, subscript 0 denotes conditions at the wall and subscript δ denotes conditions at the edge of the boundary layer. Also, prime will denote d/ds.

The approximation for the tangential velocity component in the boundary layer is

$$u(s, n) = U\{a(s)n + b(s)n^2 + c(s)n^3 + d(s)n^4\} \tag{6}$$

The boundary conditions are

$$u(s, \delta) = U(s); \qquad (\partial u/\partial n)_\delta = (\partial^2 u/\partial n^2)_\delta = 0 \tag{7a}$$

$$u(s, 0) = 0; \qquad v(s,0) = v_0(s) \tag{7b}$$

Eq. (1) and boundary conditions (7b) yield

$$v_0(\partial u/\partial n)_0 = UU' + \nu(\partial^2 u/\partial n^2)_0 \tag{8}$$

The four conditions (7a) and (8) allow the evaluation of the four coefficients in expression (6). They are

$$\begin{aligned} a &= (N+12)/\delta(M+6) & b &= 3(2M-N)/\delta^2(M+6) \\ c &= (-8M+3N-12)/\delta^3(M+6) & d &= (3M-N+6)/\delta^4(M+6) \end{aligned} \tag{9}$$

where

$$M = \delta v_0/\nu \text{ and } N = (\delta^2/\nu)U' \tag{10}$$

All coefficients in the quartic polynomial for u contain the expression $(M+6)$ in the denominator. This means that if the suction velocity reaches a value of $v_0 = -6\nu/\delta$, the velocity, $u(s, n)$, becomes infinite. However, the value of k in the denominator $(M+k)$ varies with the number of terms used in the polynomial for $u(s, n)$. For two terms $k = 2$, for three terms $k = 4$, for four terms $k = 6$, etc. Therefore, the above limitation on the velocity profile depends on the degree of approximation and has no physical significance. No such limitation

of the Pohlhausen method occurs for solid boundary or for fluid injection, since the denominator of the coefficients does not vanish for these cases.

The value of $(\partial u/\partial n)_0$ is given by the first coefficient of the polynomial (6). Expression (9) shows that this coefficient becomes zero when $N = -12$. N is proportional to δ^2 and to U'. Also, the suction parameter, M, does not appear in the numerator of a. Thus, if the boundary layer thickness is kept constant, the flow would separate independently of the suction intensity for a certain value of U'. This condition does not represent physical fact, and is a consequence of the use of Pohlhausen's original boundary conditions.

It is thought that the limitations of the Pohlhausen method are inherent in the analysis and are due to the use of three or more boundary conditions at the edge of the boundary layer and only one condition at the wall. If additional boundary conditions are needed, Pohlhausen suggested the use of the conditions that $(\partial^3 u/\partial n^3)_\delta$, $(\partial^4 u/\partial n^4)_\delta$, etc. should vanish. However, boundary conditions related to the tangential velocity and its derivatives at the edge of the boundary layer cannot be realized exactly, since u approaches the value of U asymptotically only.

In summary of this method it should be noted that for the case of boundary layer control by continuous surface suction, no agreement can be obtained between its results and reality. The results erroneously show that for a constant boundary layer thickness the location of the point of separation is not influenced by the application of suction. Also, a remarkable limitation on the magnitude of the suction velocity results which limitation has no foundation in physical phenomena.

The new method. In the new method, the von Kármán integral principle and the series approximation for the distribution of the tangential velocity in the boundary layer are preserved, but new boundary conditions are introduced. Boundary conditions at the surface are more desirable than at the edge of the boundary layer, since the effect of viscosity is the greatest at the surface. Howarth [3] noted that further boundary conditions may be found by differentiating the boundary layer momentum equation with respect to n and evaluating the derivatives at the wall. To the knowledge of the author, however, the importance of the conditions at the surface have not been fully discussed before. In evaluating the momentum equation and its derivatives at the wall, it is found that the coefficient of n^2 has to be zero when applying the method to analyses of boundary layers along solid walls (this means no suction or injection).* If in this case the use of a four term polynomial is required, a term containing n^5 should be used.

The new boundary conditions are: At the edge of the boundary layer, u should be equal to U, and $\partial u/\partial n$ should vanish. At the surface, u should either be zero or have a certain magnitude (slip velocity), and v should be equal to the suction velocity, v_0. The value of the momentum equation and its first derivative with respect to n, at the wall, yield two more boundary conditions. If higher than fourth powers of n are required in the series approximation of u, additional bound-

* This refers to all analyses where a polynomial similar to eq. (6) is used for u.

ary conditions can be obtained by evaluating the second and higher derivatives of the momentum equation at the wall.

The new boundary conditions can be expressed as

$$u(s, \delta) = U(s); \qquad (\partial u/\partial n)_\delta = 0 \tag{11a}$$

$$u(s, 0) = 0; \qquad v(s, 0) = v_0(s) \tag{11b}$$

Eq. (1), its derivative with respect to n, and boundary condition (11b) yield at the wall

$$v_0(\partial u/\partial n)_0 = UU' + \nu(\partial^2 u/\partial n^2)_0 \tag{12a}$$

$$v_0(\partial^2 u/\partial n^2)_0 = \nu(\partial^3 u/\partial n^3)_0 \tag{12b}$$

Conditions (11a), (12a), and (12b) allow the determination of the coefficients of the quartic polynomial for u. These are

$$a = (24 + 6N + MN)/D\delta, \qquad b = 3(-3N + 4M)/D\delta^2,$$

$$c = (-3MN + 4M^2)/D\delta^3, \qquad d = (-6 + 3N + 2MN - 6M - 3M^2)/D\delta^4,$$

$$D = 18 + 6M + M^2 \tag{13}$$*

Since the denominator of the coefficients, D, has no real roots, the velocity profile, $u(s, n)$, does not become infinite for any value of the suction velocity. Also, the numerator of a which determines the location of the point of separation, i.e. the point where $(\partial u/\partial n)_0$ becomes zero, depends on the suction velocity distribution as well as on the pressure gradient in the potential flow.

Use of the coefficients (13) in eq. (6) enables the evaluation of the terms in the von Kármán integral equation (5).

The following notation is introduced:

$$\left.\begin{aligned}
B_1 &= (9180N^2 + 7128N - 175392) + M(7290N^2 - 10584N - 181728) \\
&\quad + M^2(2112N^2 - 11388N - 86112) \\
&\quad + M^3(291N^2 - 3276N - 21936) + M^4(19N^2 - 416N - 3768) \\
&\quad - M^5(26N + 432) - 24M^6 \\
B_2 &= (11016N + 7128) + M(9126N - 6156) + M^2(3114N - 8064) \\
&\quad + M^3(513N - 3048) + M^4(38N - 546) - 39M^5 \\
B_3 &= (1890N^2 + 26568N + 19008) + M(2898N^2 + 19944N + 12672) \\
&\quad + M^2(909N^2 + 1368N - 19008) + M^3(76N^2 - 780N - 4032) \\
&\quad - M^4(78N + 144) \\
B_4 &= 72(17N^2 + 211N + 1778) + 6M(101N^2 + 1007N + 10838) \\
&\quad + 2M^2(38N^2 + 321N + 9108) + 3M^3(11N + 936) + 216M^4 \\
B_5 &= 6N + 24 + MN
\end{aligned}\right\} \tag{14}$$

* It can be seen that for solid wall ($M = 0$), the coefficient of the n^3 term (c) becomes zero in this analysis.

With the expressions of (14), the von Kármán integral equation yields the following differential equation for the boundary layer thickness variation and the suction or injection velocity distribution:

$$m' = (1260D^3\nu/U\delta^3B_3)\{N(B_4/1260D^2 - 1) + M + B_4\epsilon/D\} + 2U''\delta B_2/\nu B_3 + 6\delta' B_1/\delta^2 B_3 \quad (15)$$

where $m = v_0/\nu$.

Eq. (15) is the general equation describing the variation of the boundary layer thickness and the corresponding variation of the suction velocity distribution. If it is required that the thickness of the boundary layer be kept constant from a certain point on, then, for this region, the last term of eq. (15) will be zero.

For constant boundary layer thickness, eq. (15) can be written as

$$m' = G_1(\nu/U\delta^3) + G_2(U''\delta/\nu) \quad (16)$$

where

$$G_1 = (1260D^3/B_3)\{N(B_4/1260D^2 - 1) + M + \epsilon B_4/D\}, \qquad G_2 = 2B_2/B_3 \quad (17)$$

Eq. (16) can be solved by numerical or graphical methods. In the presented illustrative example, the isocline method was used.

Numerical example. Fig. 1 shows the resultant velocity distribution along a streamline for a non-swept wing of tapered planform and of constant airfoil section. The streamline is located at 75% of the semi-span, and the resultant potential flow was determined at an angle of attack of 8° and a lift coefficient of 1.01 for a free stream velocity of 150 ft/sec. A three-dimensional thin wing theory was used. The equation of the mean line was taken as

$$y/c = 0.0422(1 - 4x^2/c^2)(0.5 + x/c)$$

where c is the chord, and x, y are cartesian coordinates in the direction of the chord and perpendicular to it.

In order to obtain finite values for the velocity in the potential flow, at the leading edge an NACA 1408 airfoil nose section was used.

Fig. 2 shows the suction velocity distribution necessary to maintain a laminar boundary layer of prescribed thickness. It should be noted here that suction was started at the leading edge where the minimum thickness of the boundary layer occurred. This minimum thickness of 2.265×10^{-3} ft. was maintained over this section of the wing. Based on the free stream velocity and on the boundary layer thickness, the Reynolds number at the leading edge is 2170.

The presented example was chosen to give a severe adverse pressure gradient in the potential flow. It illustrates the character of the suction velocity distribution satisfactorily. It can be seen that relatively high suction velocities have to be used locally if the change in pressure is sudden. Once the effect of the adverse pressure gradient is counteracted, a small suction velocity is sufficient for the maintenance of a thin laminar boundary layer so that separation is prevented. Not the high local values of the suction velocity, but the area under the curve in Fig. 2 is significant for the suction-power requirement.

In other cases there will be a less severe adverse pressure gradient present at the leading edge and another region of adverse pressure gradient will exist towards the trailing edge. For such airfoils, two smaller peaks of the suction velocity will exist.

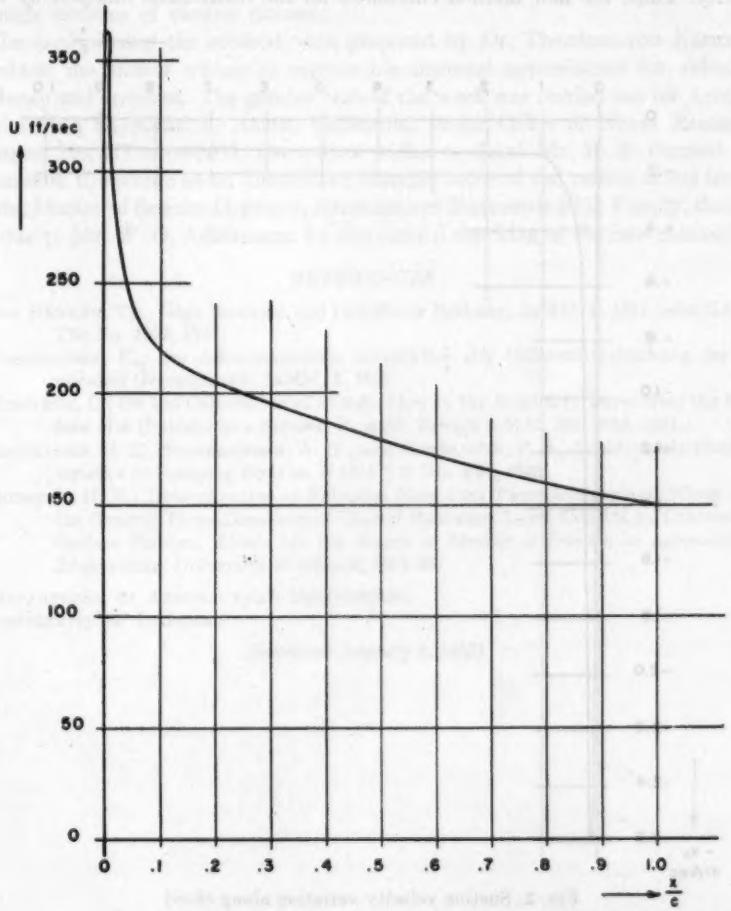

FIG. 1. Velocity variation along chord in potential flow

Calculations show that the new method eliminates the restriction on the suction velocity distribution which occurred when the modified Pohlhausen method was used. They also show that the intensity of the suction velocity is greatly affected by the pressure variation in the potential flow. This could be expected from consideration of the physical phenomena. Also, there is no constant value

of the suction velocity for which the tangential velocity in the boundary layer will be infinitely large. Lastly, as can be seen from expression (13), the derivative of the tangential velocity in the boundary layer with respect to the normal coordinate, at the wall, depends on both the pressure gradient and the suction velocity. Thus, the new method eliminates all the restrictions imposed by the

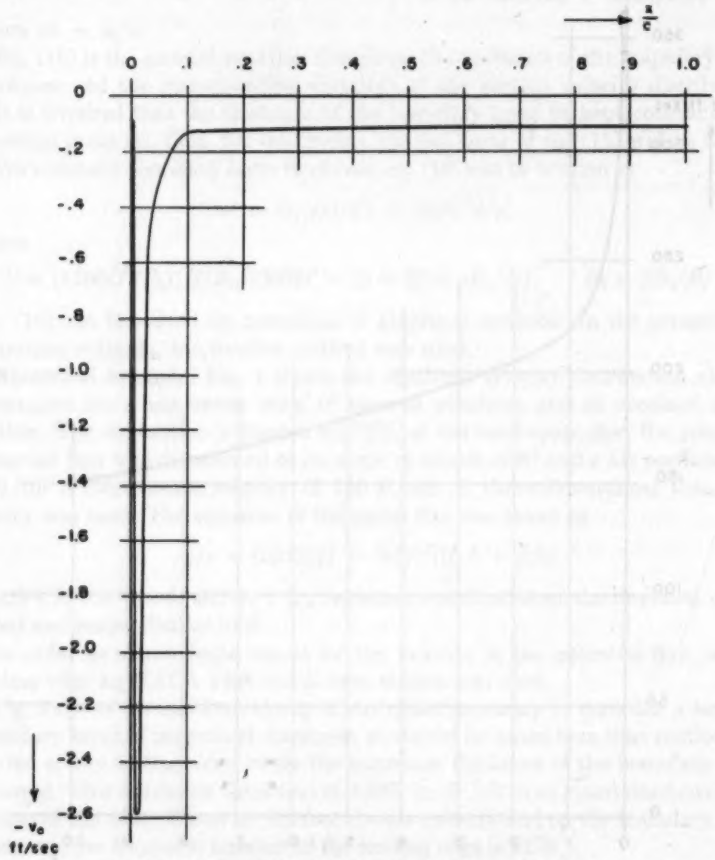

FIG. 2. Suction velocity variation along chord

Pohlhausen method on the analysis of boundary layer control by continuous surface suction.

It should be remarked that the so-called non-slip condition will be approximately true only for porous walls. It is thought that the inclusion of a slip velocity at the wall, proportional to the derivative of the suction or the injection velocity, would increase the accuracy of the results. However, such a correction of the non-slip condition makes the calculations much more cumbersome. In

addition, the correction would be of a small order of magnitude. Therefore, the non-slip condition was maintained in the present analysis, but a correction for the shear stress at the wall has been introduced in the form of a modifying function. The type and the magnitude of this modifying function should be determined from experimental results, e.g. from Schubauer's experiments [4] of flow through surfaces of various porosity.

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ON NON-UNIFORM TORSION OF CYLINDRICAL RODS*

By Eric Reissner

1. Introduction. In the following we propose to discuss briefly the problem of determining corrections to the St. Venant theory of torsion of cylindrical rods, induced by such effects as restraint against warping at the end section of the rod. The problem has previously been considered by various authors. We may mention in particular Timoshenko's work on beams with thin-walled open section [1] and with solid rectangular section [2], A. Föppl's work for a beam with elliptical cross section [3] and H. Reissner's work on torsion of thin-walled box beams [4].

We shall show how a variational theorem for stresses *and* displacements which has recently been considered by the present author [5] may be used for the approximate determination of such corrections.

2. The variational theorem. We define a function F by

$$F = \frac{\partial u}{\partial x}\sigma_x + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_{xy} + \cdots - W(\sigma_x, \cdots, \tau_{yz}) \tag{1}$$

where σ_x, τ_{xy}, $\cdots$ are stresses in the usual notation and u, v, w are components of displacements. The theorem states that the variational equation

$$\delta\left\{\iiint F\,dV - \iint_{S_1}(\bar{p}_x u + \bar{p}_y v + \bar{p}_z w)\,dS - \iint_{S_2}[p_x(u-\bar{u}) + p_y(v-\bar{v}) + p_z(w-\bar{w})]\,dS\right\} = 0 \tag{2}$$

where both stresses and displacements are varied independently is equivalent to the system of differential equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \text{ etc.} \tag{3}$$

$$\frac{\partial u}{\partial x} = \frac{\partial W}{\partial \sigma_x}, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial W}{\partial \tau_{xy}}, \text{ etc.} \tag{4}$$

and to the boundary conditions

$$p_x = \bar{p}_x, \text{ etc. on } S_1, \qquad u = \bar{u}, \text{ etc. on } S_2 \tag{5}$$

where

$$p_x = \sigma_x \cos(n, x) + \tau_{xy}\cos(n, y) + \tau_{xz}\cos(n, z), \text{ etc.} \tag{6}$$

The theorem was stated in [5] under the assumption that $\delta u = \delta v = \delta w = 0$ on S_2. Subsequently it was stated without this restriction by Fraeys de Veubeke [6] who also pointed out its significance as a canonical principle in the sense of Hamilton and Jacobi.

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3. The boundary value problem of the twisted rod. Let x, y be coordinates in the cross section of the rod while the direction of z coincides with the axis of the rod. Let $f(x, y) = 0$ be the equation of the boundary of the cross section and $z = 0$ and $z = \zeta$ coordinates of the ends of the beam.

Assume that the material of the rod is isotropic and obeys the generalized Hooke's Law. Assume that the end $z = 0$ of the rod is fixed, that the end $z = \zeta$ is free of normal stress and acted upon by a prescribed twisting couple $\overline{T}$ and that the walls $f(x, y) = 0$ of the rod are free of stress.

With these assumptions the variational equation (2) becomes

$$\delta\left\{\iiint\left[\frac{\partial u}{\partial x}\sigma_x + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_{xy} + \cdots - \frac{\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2}{2G}\right.\right.$$
$$\left.\left. - \frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z)}{2E}\right] dV \qquad (7)$$
$$\left. - \iint_{z=\zeta}(\bar{\tau}_{xz}u + \bar{\tau}_{yz}v)\,dx\,dy + \iint_{z=0}(\tau_{xz}u + \tau_{yz}v + \sigma_z w)dx\,dy\right\} = 0$$

where

$$\iint(\bar{\tau}_{xz}y - \bar{\tau}_{yz}x)\,dx\,dy = \overline{T}$$
$$\iint \bar{\tau}_{xz}\,dx\,dy = \iint \bar{\tau}_{yz}\,dx\,dy = 0 \qquad (8)$$

4. Assumptions for an approximate solution. In applying the variational equation (7) we shall make the assumption that to the degree of approximation here contemplated we may set as in St. Venant's theory

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \qquad (9)$$

We also assume that we may carry over St. Venant's result that each cross section moves without distortion. Assuming for simplicity's sake that the cross section of the beam is symmetric with respect to the x and y-axis this means that we set

$$u = \beta(z)y, \qquad v = -\beta(z)x \qquad (10)$$

where $\beta(z)$ is the angle of rotation of the sections $z = \text{const}$. In St. Venant's theory we have $\beta(z) = cz$ but this result is *not* retained here.

We further assume that our approximations for displacements satisfy the boundary conditions imposed on displacements,

$$\beta(0) = 0, \qquad w(x, y, 0) = 0 \qquad (11)$$

Introduction of equations (9) to (11) into the variational equation (7) reduces this equation to the form

$$\delta\left\{\iiint\left[\sigma_z\frac{\partial w}{\partial z} + \left(\beta' y + \frac{\partial w}{\partial x}\right)\tau_{xz} + \left(-\beta' x + \frac{\partial w}{\partial y}\right)\tau_{yz}\right.\right.$$
$$\left.\left. - \frac{\sigma_z^2}{2E} - \frac{\tau_{xz}^2 + \tau_{yz}^2}{2G}\right] dx\,dy\,dz - \beta(\zeta)\overline{T}\right\} = 0 \qquad (12)$$

where primes indicate differentiation with respect to z.

We could now apply equation (12) without making further simplifying assumptions. We prefer, however, to discuss instead the effect of a number of different additional simplifying assumptions.

It may be worth noting at this point that we have been able to make approximations to stresses and displacements simultaneously while in the application of either the minimum principle for displacements or the minimum principle for stresses in elasticity we are required to be consistent in making approximations for either displacements or stresses. (For a discussion of this aspect of the problem in the theory of bending of plates see [5].)

5. Additional approximation I. We set, again guided by St. Venant's torsion theory.

$$w = \beta'(z)\phi(x, y) \tag{13}$$

When ϕ is the warping function of St. Venant's theory then the only modification of St. Venant's theory will be the generalization of St. Venant's factor cz into $\beta(z)$, that is a shift from uniform twist for no axial restraint to non-uniform twist caused by axial restraint. For our purposes ϕ need not be St. Venant's warping function, which will henceforth be designated by ϕ_{SV}, but merely an approximation to ϕ_{SV}. However, when $\phi = \phi_{SV}$ certain interesting simplifications of the results will appear.

In view of (11) we now have as condition of no warping at the fixed end of the beam

$$\beta'(0) = 0 \tag{14}$$

and the variational equation (12) becomes

$$\delta\left\{\iiint\left[\sigma_z\,\phi\,\beta'' + (y + \phi_x)\beta'\,\tau_{xz} + (-x + \phi_y)\beta'\,\tau_{yz} - \frac{\sigma_z^2}{2E} - \frac{\tau_{xz}^2 + \tau_{yz}^2}{2G}\right]dx\,dy\,dz - \beta(\zeta)\overline{\mathrm{T}}\right\} = 0 \tag{15}$$

We carry out the variations in (15), integrate suitably by parts and take account of the boundary conditions (11) and (14). We then have

$$\begin{aligned}
&\iiint\left[\{\phi\beta'' - (\sigma_z/E)\}\,\delta\sigma_z + \{(y + \phi_x)\beta' - (\tau_{xz}/G)\}\,\delta\tau_{xz}\right.\\
&\qquad\left. + \{(-x + \phi_y)\,\beta' - (\tau_{yz}/G)\}\,\delta\tau_{yz}\right]dx\,dy\,dz\\
&\quad - \int\left[\iint\left((\partial/\partial z)\,\{\tau_{xz}\,(y + \phi_x) + \tau_{yz}\,(-x + \phi_y)\}\right.\right.\\
&\qquad\left.\left. - (\partial^2/\partial z^2)\,\{\sigma_z\,\phi\}\right)dx\,dy\right]\delta\beta\,dz + \left[\iint\sigma_z\,\phi\,dx\,dy\,\delta\beta'\right]_\zeta\\
&\quad + \left[\iint\{\tau_{xz}\,(y + \phi_x) + \tau_{yz}\,(-x + \phi_y)\}\,dx\,dy\right.\\
&\qquad\left. - \iint\{\partial(\sigma_z\,\phi)/\partial z\}\,dx\,dy - \overline{\mathrm{T}}\right]_\zeta\delta\beta(\zeta) = 0
\end{aligned} \tag{16}$$

Equation (16) implies the following relations

$$\sigma_z = E\phi\beta'', \qquad \tau_{zx} = G(y + \phi_x)\beta', \qquad \tau_{yz} = G(-x + \phi_y)\beta'. \tag{17}$$

$$\iint [\tau_{zx}(y + \phi_x) + \tau_{yz}(-x + \phi_y) - (\partial\sigma_z/\partial z)\phi]\, dx\, dy = \overline{T} \tag{18}$$

$$\left[\iint \sigma_z\, \phi\, dx\, dy\right]_\zeta = 0 \tag{19}$$

Equation (19) together with the first of equations (17) supplies the following further boundary conditions for β,

$$\beta''(\zeta) = 0 \tag{20}$$

Combination of (17) and (18) results in the desired differential equation for the variable angle of twist β.

We set as abbreviations

$$\iint \phi^2\, dx\, dy = \Gamma, \qquad \iint (\phi_x^2 + \phi_y^2)\, dx\, dy = D,$$
$$\iint (x\phi_y - y\phi_x)\, dx\, dy = D^*, \qquad \iint (x^2 + y^2)\, dx\, dy = I_p \tag{21}$$

and have then for this differential equation

$$G[I_p + D - 2D^*]\beta' - E\Gamma\beta''' = \overline{T} \tag{22}$$

When $\phi = \phi_{SV}$ equation (22) may be brought into a simpler form as follows. The boundary condition for ϕ_{SV}, $\partial\phi_{SV}/\partial n = -\frac{1}{2}\partial(x^2 + y^2)/\partial s$, is responsible, as was noted by Weinstein [7], for the equality

$$D_{SV}^* = D_{SV} \tag{23}$$

Furthermore $I_p - D_{SV}$ is the torsional rigidity C of the beam according to St. Venant's theory. Consequently, when $\phi = \phi_{SV}$ we have

$$GC\beta' - E\Gamma\beta''' = \overline{T} \tag{24}$$

Equation (24) has been given by Wagner [8] and was also derived by Goodier by a different variational approach [9].

We may make the following observation. By postulating that the amplitude of the warping displacement w in (13) equals the derivative of the angle of rotation β we also postulate that at a section where warping is prevented we have vanishing shearing strains γ_{zx} and γ_{yz}. This is certainly not the case in reality and should in general not be assumed, even in first approximation since these shearing strains are proportional to the primary shearing stresses τ_{zx} and τ_{yz}.

6. Additional approximation II. We now replace equation (13) for the warping displacement w by

$$w = \alpha(z)\phi(x, y) \tag{25}$$

and use the variational equation (12) for the purpose of obtaining two simultaneous differential equations for α and β.

Equation (12) now becomes

$$\delta\left\{\iiint[\sigma_z\phi\alpha' + \tau_{zz}(y\beta' + \phi_x\alpha) + \tau_{yz}(-x\beta' + \phi_y\alpha) - (\sigma_z^2/2E) - (\tau_{zz}^2 + \tau_{yz}^2)/2G]\,dx\,dy\,dz - \beta(\zeta)\mathrm{T}\right\} = 0 \tag{26}$$

while the condition of no warping at the fixed end of the beam is now

$$\alpha(0) = 0 \tag{27}$$

Proceeding as before we now obtain instead of (17), (18) and (19)

$$\sigma_z = E\phi\alpha', \qquad \tau_{zz} = G(y\beta' + \phi_x\alpha), \qquad \tau_{yz} = G(-x\beta' + \phi_y\alpha) \tag{28}$$

$$\iint[\tau_{zz}\phi_x + \tau_{yz}\phi_y - [\partial\sigma_z/\partial z)\phi]\,dx\,dy = 0 \tag{29}$$

$$\iint[\tau_{zz}y - \tau_{yz}x]\,dx\,dy = \overline{\mathrm{T}} \tag{30}$$

$$\left[\iint\sigma_z\phi\,dx\,dy\right]_\zeta = 0 \tag{31}$$

Equations (31) and (28) supply as boundary condition which takes the place of (20)

$$\alpha'(\zeta) = 0 \tag{32}$$

Combination of (28), (29) and (30) and use of the abbreviations (21) results in the following two differential equations for α and β

$$G[-D^*\beta' + D\alpha] - E\Gamma\alpha'' = 0 \tag{33}$$

$$G[I_p\beta' - D^*\alpha] = \overline{\mathrm{T}} \tag{34}$$

Equations (33) and (34) may be reduced to one equation for β, of the form

$$G\left(I_p - \frac{D^{*2}}{D}\right)\beta' - E\Gamma\frac{I_p}{D}\beta''' = \overline{\mathrm{T}} \tag{35}$$

while condition (27) for the absence of warping at the fixed end becomes

$$\beta'(0) = \frac{\overline{\mathrm{T}}}{GI_p} \tag{36}$$

When $\phi = \phi_{SV}$ and therewith $D^* = D = I_p - C$ we have from (35)

$$GC\beta' - \frac{E\Gamma}{I - (C/I_p)}\beta''' = \overline{\mathrm{T}} \tag{37}$$

Equation (37) and (36) may be compared with (24) and (14) of the preceding section. We see that now β' no longer vanishes at the fixed end of the beam. In addition we have obtained a differential equation for β in which one of the

coefficients differs from the corresponding coefficient which was obtained on the basis of the assumption that $\alpha = \beta'$.

The condition (32) of no normal stress at the free end becomes with (34)

$$\beta''(\zeta) = 0 \tag{38}$$

and in this point the results of the present section agree with the results of the preceding section.

7. Additional approximation III. Instead of making as before an assumption for the warping displacement w we now make an assumption for the warping stress σ_z. Since, as long as $\sigma_x = \sigma_y = 0$, we have $E\partial w/\partial z = \sigma_z$ we now set [2]

$$\sigma_z = \sigma(z)\phi(x, y) \tag{39}$$

We require that this expression for σ_z satisfies the condition of no normal stress at the free end of the beam by prescribing

$$\sigma(\zeta) = 0 \tag{40}$$

We shall find that we cannot now prescribe that the approximate warping displacement w which we obtain by the variational procedure vanishes at every point of the fixed end $z = 0$. Rather, the variational equation provides an appropriate approximate form of this boundary condition.

Introducing (8), (9), (10), (39) and the first of equations (11) into the variational equation (7) we have

$$\delta\left\{\iiint\left[\sigma\phi\frac{\partial w}{\partial z} + \left(\beta' y + \frac{\partial w}{\partial x}\right)\tau_{zx} + \left(-\beta' x + \frac{\partial w}{\partial y}\right)\tau_{yz} - \frac{\sigma^2\phi^2}{2E} - \frac{\tau_{zx}^2 + \tau_{yz}^2}{2G}\right]dx\,dy\,dz - \beta(\zeta)\,\mathrm{T} + \sigma(0)\iint\phi w_0\,dx\,dy\right\} = 0 \tag{41}$$

The following system of relations is obtained from (41)

$$\iint\left(\phi\frac{\partial w}{\partial z} - \phi^2\frac{\sigma}{E}\right)dx\,dy = 0, \tag{42}$$

$$\beta' y + \frac{\partial w}{\partial x} - \frac{\tau_{zx}}{G} = 0, \qquad \beta' x + \frac{\partial w}{\partial y} - \frac{\tau_{yz}}{G} = 0. \tag{43}$$

$$\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \sigma'\phi = 0 \tag{44}$$

$$\frac{\partial}{\partial z}\iint(\tau_{zx}y - \tau_{yz}x)\,dx\,dy = 0 \tag{45}$$

$$z = \zeta: \qquad \iint(\tau_{zx}y - \tau_{yz}x)\,dx\,dy = \mathrm{T} \tag{46}$$

$$f(x, y) = 0: \qquad \tau_{zx}\,dy - \tau_{yz}\,dx = 0 \tag{47}$$

$$z = 0: \qquad \iint\phi w\,dx\,dy = 0 \tag{48}$$

In order to reduce the system (42) to (48) to two simultaneous ordinary differential equations for β and σ we proceed as follows.

Introduction of (43) into (44) and (47) leads to a boundary value problem for w,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\sigma' \frac{\phi}{G} \tag{49}$$

$$f(x, y) = 0: \qquad \frac{\partial w}{\partial n} = -\beta' \frac{\partial}{\partial s}\left(\frac{x^2 + y^2}{2}\right)$$

In (49) we set

$$w = -\frac{\sigma'}{G}\psi + \beta' \chi \tag{50}$$

where

$$\nabla^2 \psi = \phi, \qquad \nabla^2 \chi = 0$$

$$f(x, y) = 0: \qquad \frac{\partial \psi}{\partial n} = 0, \qquad \frac{\partial \chi}{\partial n} = -\frac{\partial}{\partial s}\left(\frac{x^2 + y^2}{2}\right) \tag{51}$$

We note that $\chi = \phi_{sv}$ and that ψ may be obtained by the same method as ϕ_{sv}.

Introduction of (50) into (42) gives

$$-\frac{\sigma''}{G}\int \phi\psi \, dS + \beta'' \int \phi\chi \, dS - \frac{\sigma}{E}\int \phi^2 \, dS = 0 \tag{52}$$

which is one of the two equations for σ and β.

Introduction of (50) into (43) and into (45) and (46) gives as the other of the two simultaneous equations,

$$G\beta' I_p - \sigma' \int \left(\frac{\partial \psi}{\partial x} y - \frac{\partial \psi}{\partial y} x\right) dS + G\beta' \int \left(\frac{\partial \chi}{\partial x} y - \frac{\partial \chi}{\partial y} x\right) dS = \mathrm{T} \tag{53}$$

Equations (52) and (53) may be reduced, as in the preceding approximations, to one third order equation for β.

Equations (50) and (48) give as boundary condition at the fixed end.

$$-(1/G)\sigma'(0) \int \phi\psi \, dS + \beta'(0) \int \phi\chi \, dS = 0 \tag{54}$$

Equations (52) to (54) may be written in somewhat more compact form with the use of the abbreviations (21) and by making use of certain identities from potential theory. We shall carry out this transformation under the assumption that $\phi = \chi = \phi_{sv}$.

We then have

$$\int \phi\chi \, dS = \Gamma, \qquad \int \left(\frac{\partial \chi}{\partial x} y - \frac{\partial \chi}{\partial y} x\right) dS = -D \tag{55}$$

and

$$\int \left(\frac{\partial \psi}{\partial x} y - \frac{\partial \psi}{\partial y} x\right) dS = \oint \psi(y \, dy + x \, dx) = -\oint \psi \frac{\partial \phi}{\partial n} \, ds \tag{56a}$$

Application of Green's identity

$$\int (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dS = \oint \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds$$

furnishes on the basis of equations (51)

$$\oint \psi \frac{\partial \phi}{\partial n}\, ds = - \int \phi \nabla^2 \psi\, dS = - \int \phi^2\, dS = - \Gamma \tag{56b}$$

There remains

$$\begin{aligned} \int \phi\psi\, dS &= \int \psi \nabla^2 \psi\, dS \\ &= - \oint \psi \frac{\partial \psi}{\partial n}\, dS - \int \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dS = - D(\psi) \end{aligned} \tag{57}$$

Equations (52) to (54) read now

$$(1/G) D(\psi) \sigma'' + \Gamma \beta'' - (1/E) \Gamma \sigma = 0 \tag{58}$$

$$G(I_p - D)\beta' - \Gamma \sigma' = \bar{T} \tag{59}$$

$$(1/G) D(\psi) \sigma'(0) + \Gamma \beta'(0) = 0 \tag{60}$$

Elimination of σ gives

$$GC\beta' - E \frac{\Gamma^2 + D(\psi) C}{\Gamma} \beta''' = \bar{T} \tag{61}$$

and

$$\beta'(0) = \frac{D(\psi)}{\Gamma^2 + D(\psi) C} \frac{\bar{T}}{G} \tag{62}$$

A noteworthy observation concerning the result of approximation III is the following. While in approximation II the warping displacement $w = \alpha\phi$ was of the same form all along the axis of the rod we now have an expression for w containing two different functions of x, y, namely ϕ and ψ. This fact makes it probable that the present approximation III is closer to the true result than the preceding approximation II.

We also note that approximation III permits the calculation of the warping stress σ_z without differentiation once β and σ are known, while in approximation II $\sigma_z = E\, \partial w / \partial z$ is determined by differentiation of $\alpha(z)$.

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ON A CLASS OF ITERATIVE PROCEDURES FOR SOLVING NORMAL SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS*

By J. Warga

1. Introduction. Let

$$(A) \qquad \dot{z} = f(z, t), \qquad z(t_0) = z_0,$$

where z and $f(z, t)$ are n-dimensional vectors, be a system of ordinary differentia equations with given initial conditions.

The investigation of iterative procedures for constructing solutions of system (A) has been rather sketchy and has left many questions unanswered. These procedures are of great interest both in theoretical work and in practical calculations. Various such procedures were devised and applied by Cauchy and Picard in connection with existence theorems; by Liapunov and Poincaré, in their investigations of stability; by Lichtenstein, in discussing the dependence of solutions on initial conditions and parameters. In applied mathematics, they are a powerful tool for "refining" approximate solutions obtained by other means.

Various iterative procedures were introduced for the most part unrelated to one another. While their convergence to the exact solution of system (A) was in each case proven, little attention was paid, in general, to the form and rapidity of this convergence.

We are concerned here essentially with three problems relating to the iterative procedures. First of all, we are interested in finding a unified framework incorporating their common features and permitting to study some of their convergence properties simultaneously. In the second place, we intend to discuss a little more in detail the rapidity of convergence of particular procedures. Finally, we introduce iterative procedures designated as the "A-procedure" and the "iterative k-th degree procedures". The iterative k-th degree procedures are designed for numerical work.

We define, in §2, a class I of iterative procedures and prove their convergence to the solution of (A). In §3, we indicate a number of iterative procedures described in the literature and show that they belong to I. In §4, we describe the suggested procedures which likewise belong to I. In §5, we indicate certain general properties of convergence of I and study those which characterize particular procedures. Finally, in the Appendix, we consider generalizations of the procedures of I to integral and functional equations.

The letters z, y, u, v, $f(z, t)$, G_j, H_j represent n-dimensional real vectors. The letter t represents a real valued independent variable. The norm of a vector $z(t)$, $\| z(t) \|$, represents for each value of t the largest of the absolute values of the components of $z(t)$. The norm of a matrix $A = (a_{ij})$, $\| A \|$, is the largest of the expressions $\sum_j | a_{ij} |$.

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2. The Class I of Iterative Procedures. *Definition 1.* The (vector) function $f(u, t)$ is said to have property $P_A(z_0)$ if there exist constants a, b, K_f and M_f such that

1) $f(u, t)$ is continuous in u and t for $| t - t_0 | \leqq a$, $\| u - z_0 \| \leqq b$

2) $\| f(u, t) \| \leqq M_f$ for $| t - t_0 | \leqq a$, $\| u - z_0 \| \leqq b$

3) $f(u, t)$, as a function of u, satisfies a Lipschitz condition with the constant K_f for $| t - t_0 | \leqq a$, $\| u - z_0 \| \leqq b$, that is,

$$\| f(Y, t) - f(y, t) \| \leqq K_f \| Y - y \| \text{ for } | t - t_0 | \leqq a,$$
$$\| Y - z_0 \| \leqq b, \qquad \| y - z_0 \| \leqq b.$$

We shall apply the following well known

EXISTENCE THEOREM A: [1]* If the function $f(u, t)$ has the property $P_A(z_0)$, then the system (A) has a unique solution $z(t)$ for $| t - t_0 | \leqq \text{Min } (a, bM_f^{-1})$ and we have:

$$\| z(t) - z_0 \| \leqq b, \| \dot{z}(t) \| \leqq M_f \text{ for } | t - t_0 | \leqq \text{Min } (a, bM_f^{-1}).$$

Let us now consider the system (A) in which $f(u, t)$ is assumed to have property $P_A(z_0)$. We define a sequence of (vector) functions $y_j(t)$, $j = 0, 1, 2 \cdots$, as follows: Let $y_0(t)$ be a continuous function of t, such that

$$y_0(t_0) = z_0 \text{ and } \| y_0(t) - z_0 \| \leqq b \text{ for } | t - t_0 | \leqq \text{Min } (a, bM_f^{-1}).$$

Let $y_j(t), j \geqq 1$, be the solution of the system

$$\text{(I)} \qquad \dot{y}_j = G_j(y_j, t), \qquad y_j(t_0) = z_0,$$

where the (vector) functions $G_j(u, t)$ of the (vector) variable u and of t satisfy the following conditions (henceforth designated as "the weak property $P_I(f)$"):

1) $\| G_j(u, t) \| \leqq M_G$ in the domain $\bar{D}$: $| t - t_0 | \leqq \text{Min } (a, bM_f^{-1}, bM_G^{-1})$, $\| u - z_0 \| \leqq b$, where M_G is some constant independent of j.

2) $G_j(u, t)$ is continuous in u and t in the domain $\bar{D}$.

3) $G_j(u, t)$ as a function of u, satisfies for all j a Lipschitz condition with the constant K_G in $\bar{D}$.

4) $G_{j+1}(y_j(t), t) - f(y_j(t), t)$ converges to 0 in the mean of order 1 for $| t - t_0 | \leqq \text{Min } (a, bM_G^{-1}, bM_f^{-1})$.

We shall say that the functions $G_j(u, t)$ have the "strong property $P_I(f)$" if condition 4) is replaced by condition

4') $G_{j+1}(y_j(t), t) - f(y_j(t), t)$ converges to 0 uniformly in t for $| t - t_0 | \leqq \text{Min } (a, bM_G^{-1}, bM_f^{-1})$.

* The numbers in brackets refer to the bibliography.

It may be observed that condition 4) (or 4')) implies a dependence of $G_{j+1}(u, t)$, hence of $y_{j+1}(t)$, on $y_j(t)$. It also establishes a link between the iterative procedure and the system (A).

We justify our definition of the sequence $y_j(t)$ in Lemma 1. We show then, in Theorem I, that the sequence $y_j(t)$ thus defined converges uniformly to the solution of the system (A) for

$$| t - t_0 | \leqq T = \text{Min } (a, bM_G^{-1}, bM_f^{-1}).$$

LEMMA 1. For all $j \geqq 1$, $y_j(t)$ exist, are differentiable and we have:

$$y_j(t_0) = z_0 \text{ and } \| y_j(t) - z_0 \| \leqq b \text{ for } | t - t_0 | \leqq T.$$

Moreover, the derivatives $\dot{y}_j(t)$ are uniformly bounded by $\pm M_G$ for $| t - t_0 | \leqq T$.

PROOF: Lemma 1 is a direct consequence of Existence Theorem A and conditions 1), 2), and 3) of $P_I(f)$.

We shall now proceed to prove

THEOREM I: Let $f(u, t)$ have property $P_A(z_0)$ and let the vector functions $G_j(u, t)$ satisfy conditions 1), 2) and 3) of $P_I(f)$.

Then the necessary and sufficient condition that

$$y_j(t) \underset{j \to \infty}{\to} z(t) \text{ uniformly in } t \text{ for } | t - t_0 | \leqq T$$

and

$$\dot{y}_j(t) \underset{j \to \infty}{\to} \dot{z}(t) \text{ in the mean of order 1 for } | t - t_0 | \leqq T$$

is condition 4);

The necessary and sufficient condition that

$$y_j(t) \underset{j \to \infty}{\to} z(t) \text{ uniformly in } t \text{ for } | t - t_0 | \leqq T$$

and

$$\dot{y}_j(t) \underset{j \to \infty}{\to} \dot{z}(t) \text{ uniformly in } t \text{ for } | t - t_0 | \leqq T$$

is condition 4').

PROOF: We shall assume in the proof of Theorem I that $t_0 = 0$, $z_0 = 0$ and $t \geqq 0$. This assumption will not affect the generality of our argument, as it corresponds only to the change of variables $Z = z - z_0$, $\bar{t} = | t - t_0 |$. We shall consider t only in the interval $(0, T)$.

Condition 4) (4')) *is sufficient.*

We first require a Lemma.

LEMMA 2. Let $k(t)$ and $h(t)$ be continuous scalar functions of t and let K be a constant.

Then

$$\text{(II)} \quad k(t) \leqq h(t) + K \int_0^t k(s)\, ds \quad \text{implies} \quad k(t) \leqq h(t) + Ke^{Kt} \int_0^t e^{-Ks} h(s)\, ds.$$

PROOF: Lemma 2 may be proven by following, with slight modifications, the argument in [2].

Let now $z(t)$ be the solution of system (A) and let

$$v_j(t) = z(t) - y_j(t),$$

where $v_j(t)$ is defined for $t \leqq T$. We have

$$\text{(III)} \qquad \begin{aligned} \dot{v}_{j+1} &= \dot{z} - \dot{y}_{j+1} = f(z, t) - G_{j+1}(y_{j+1}, t) \\ v_{j+1}(0) &= 0, \end{aligned}$$

where v_{j+1}, y_{j+1}, z stand for $v_{j+1}(t)$, $y_{j+1}(t)$, $z(t)$. Now

$$\begin{aligned} \| f(z, t) - G_{j+1}(y_{j+1}, t) \| &\leqq \| f(z, t) - f(y_j, t) \| \\ &\quad + \| f(y_j, t) - G_{j+1}(y_j, t) \| \\ &\quad + \| G_{j+1}(y_j, t) - G_{j+1}(y_{j+1}, t) \| \end{aligned}$$

Condition 3) of $P_I(f)$ implies

$$\| G_{j+1}(y_j, t) - G_{j+1}(y_{j+1}, t) \| \leqq K_G \| y_j - y_{j+1} \|$$

Condition 4) of $P_I(f)$ implies

$$\| f(y_j, t) - G_{j+1}(y_j, t) \| = \alpha_j(t),$$

where $\lim_{j \to \infty} \int_0^T \alpha_j(s)\, ds = 0$. Property $P_A(z_0)$ implies

$$\| f(z, t) - f(y_j, t) \| \leqq K_f \| z - y_j \| = K_f \| v_j \|$$

Thus

$$\begin{aligned} \| f(z, t) - G_{j+1}(y_{j+1}, t) \| &\leqq K_f \| v_j \| + K_G \| y_j - y_{j+1} \| \\ \text{(IV)} \qquad + \alpha_j(t) &\leqq K_f \| v_j \| + K_G \| y_j - z \| + K_G \| z - y_{j+1} \| \\ &\quad + \alpha_j(t) = (K_f + K_G) \| v_j \| + K_G \| v_{j+1} \| + \alpha_j(t). \end{aligned}$$

Now (III) and (IV) yield:

$$\| v_{j+1}(t) \| = \left\| \int_0^t \dot{v}_{j+1}(s) ds \right\| = \left\| \int_0^t \{ f(z(s), s) - G_{j+1}(y_{j+1}(s), s) \}\, ds \right\|$$

$$\leqq K_G \int_0^t \| v_{j+1}(s) \|\, ds + (K_f + K_G) \int_0^t \| v_j(s) \|\, ds + \int_0^t \alpha_j(s)\, ds$$

$$\leqq K_G \int_0^t \| v_{j+1}(s) \|\, ds + (K_f + K_G) \int_0^t \| v_j(s) \|\, ds + \varepsilon_j,$$

where $\varepsilon_j = \int_0^T \alpha_j(s)\, ds$ and $\varepsilon_j \to 0$ as $j \to \infty$. Clearly, $\int_0^t \| v_j(s) \|\, ds$ and

$\| v_{j+1}(t) \|$ are continuous non-negative functions of t and Lemma 2 is applicable and yields:

$$\| v_{j+1}(t) \| \leqq (K_f + K_g) \int_0^t \| v_j(s) \| \, ds + \varepsilon_j$$

$$+ K_g e^{K_g t} \int_0^t e^{-K_g s} \left\{ (K_f + K_g) \int_0^s \| v_j(s_1) \| \, ds_1 + \varepsilon_j \right\} ds.$$

Now, if $K_g \neq 0$,

$$\int_0^t e^{-K_g s} \, ds \int_0^s \| v_j(s_1) \| \, ds_1 = (1/K_g) \int_0^t e^{-K_g s} \| v_j(s) \| \, ds$$

$$- (1/K_g) e^{-K_g t} \int_0^t \| v_j(s) \| \, ds,$$

hence,

$$\| v_{j+1}(t) \| \leqq (K_f + K_g) e^{K_g t} \int_0^t e^{-K_g s} \| v_j(s) \| \, ds + K_g e^{K_g t} \varepsilon_j \int_0^t e^{-K_g s} \, ds + \varepsilon_j$$

$$= (K_f + K_g) e^{K_g t} \int_0^t e^{-K_g s} \| v_j(s) \| \, ds + e^{K_g t} \varepsilon_j .$$

This relation remains valid for $K_g = 0$ as can be easily verified. Let now $\| v_j(t) \| e^{-K_g t} = m_j(t)$ and $K_f + K_g = D$.
Then

$$m_{j+1}(t) \leqq D \int_0^t m_j(s) \, ds + \varepsilon_j .$$

Thus

$$m_1(t) \leqq \varepsilon_0 + D \int_0^t m_0(s) \, ds$$

. . . . . .

$$m_j(t) \leqq \varepsilon_{j-1} + \varepsilon_{j-2} Dt + \frac{1}{2!} \varepsilon_{j-3} D^2 t^2 + \cdots + \frac{1}{(j-1)!} \varepsilon_0 D^{j-1} t^{j-1}$$

$$+ \frac{1}{(j-1)!} D^j \int_0^t m_0(s)(t-s)^{j-1} \, ds$$

or

$$m_j(t) \leqq \sum_{r=0}^{j-1} \frac{1}{r!} \varepsilon_{j-1-r} D^r t^r + \frac{1}{(j-1)!} D^j \int_0^t m_0(s)(t-s)^{j-1} \, ds.$$

This is equivalent to

$$\text{(V)} \quad \| v_j(t) \| \leqq e^{K_g t} \sum_{r=0}^{j-1} \frac{1}{r!} \varepsilon_{j-1-r} D^r t^r + \frac{1}{(j-1)!} e^{K_g t} D^j \int_0^t m_0(s)(t-s)^{r-1} \, ds$$

We shall now show that the expression on the right hand side of the inequality sign converges to 0 uniformly in t. Indeed,

$$\frac{1}{(j-1)!}\, e^{K_0 t} D^j \int_0^t m_0(s)(t-s)^{j-1}\, ds \leqq M_0 e^{K_0 t}\, \frac{D^j t^j}{j!},$$

where $M_0 = \operatorname{Max} m_0(t) (t \leqq T)$, hence approaches 0 uniformly in t over any finite interval of the t-axis. Moreover,

$$\sum_{r=0}^{j-1} \frac{1}{r!}\, D^r t^r \varepsilon_{j-r-1} = \sum_{r=0}^{[\frac{1}{2}j]} \varepsilon_{j-r-1}\, \frac{1}{r!}\, D^r t^r + \sum_{r=[\frac{1}{2}j]+1}^{j-1},$$

where $[\frac{1}{2}j]$ = the largest integer $\leqq \frac{1}{2}j$. As $\varepsilon_s \to 0$ for large s, we may find a sufficiently large N so that $\varepsilon_{j-r-1} \leqq \kappa$ for $r \leqq [\frac{1}{2}j]$ and $j \geqq N$, where κ may be arbitrarily small.
Thus, for $j \geqq N$,

$$\sum_{r=0}^{j-1} \frac{1}{r!}\, D^r t^r \varepsilon_{j-r-1} \leqq \kappa e^{Dt} + \varepsilon \sum_{r=[\frac{1}{2}j]+1}^{\infty} \frac{1}{r!}\, D^r t^r,$$

where $\varepsilon = \operatorname{Max} \varepsilon_j$. This last expression may clearly be made arbitrarily small uniformly in t as $j \to \infty$ and we conclude that $v_j(t) \to 0$ uniformly in t as $j \to \infty$. This proves the convergence of the sequence $y_j(t)$ to $z(t)$.

We observe now that, by relation (IV),

$$\| \dot{z} - \dot{y}_{j+1} \| = \| f(z, t) - G_{j+1}(y_{j+1}, t) \| \leqq D \| v_j \| + K_0 \| v_{j+1} \| + \alpha_j(t).$$

Since $v_j(t) \to 0$ uniformly in t as $j \to \infty$, this relation shows that the sequence $\dot{y}_j(t)$ converges to $\dot{z}(t)$ in the mean of order 1 if condition 4) holds, and uniformly in t if condition 4') holds.

Condition 4) (4')) *is necessary.*

We have:

$$\| G_{j+1}(y_j, t) - f(y_j, t) \| \leqq \| G_{j+1}(y_j, t) - G_{j+1}(y_{j+1}, t) \| + \| G_{j+1}(y_{j+1}, t) - f(z, t) \| + \| f(z, t) - f(y_j, t) \|$$

By condition 3) of $P_1(f)$

$$\| G_{j+1}(y_j, t) - G_{j+1}(y_{j+1}, t) \| \leqq K_0 \| y_j - y_{j+1} \| \leqq K_0 \| v_j \| + K_0 \| v_{j+1} \|$$

Moreover,

$$G_{j+1}(y_{j+1}, t) - f(z, t) = \dot{y}_{j+1} - \dot{z},$$

since y_{j+1} satisfies system (I) and z satisfies system (A).

Finally, $\| f(z, t) - f(y_j, t) \| \leqq K_f \| z - y_j \| = K_f \| v_j \|$ by property $P_A(z_0)$.

Thus

$$\| G_{j+1}(y_j, t) - f(y_j, t) \| \leqq (K_0 + K_f) \| v_j \| + K_0 \| v_{j+1} \| + \| \dot{y}_{j+1} - \dot{z} \|,$$

which shows that condition 4) (4')) is necessary if $v_j \to 0$ uniformly in t and $\dot{v}_j \to 0$ in the mean of order 1 (uniformly in t).

This completes the proof of Theorem I.

If, in the relation (IV), we replace $\alpha_j(t)$ by δ_j, then a rather simple computation will allow to substitute for (V) the slightly stronger relation

$$\text{(V')} \qquad \| v_j(t) \| \leqq e^{K_0 t} \sum_{r=0}^{j} \frac{1}{(r+1)!} D^r t^{r+1} \delta_{j-r-1} + \frac{1}{(j-1)!} e^{K_0 t} D^j \int_0^t (t-s)^{j-1} m_0(s)\, ds$$

This last relation is valid if the iterative procedure considered has strong property $P_I(f)$.

3. Particular Iterative Procedures. An important subclass of I consists of procedures whose defining functions $G_j(u, t)$ are of the form

$$G_j(u, t) = H_j(u, y_{j-1}(t), y_{j-2}(t), \cdots, y_0(t), t).$$

To be more precise, the functions $G_j(u, t)$ are then determined at any time t by the corresponding values $y_{j-1}(t), y_{j-2}(t), \cdots, y_0(t)$. This need not always be the case. Indeed, in general, the functions $G_j(u, t)$ may be functionals of $y_k(\sigma)$, $k < j$, depending for any value of t and u on the behavior of $y_k(\sigma)$ at points σ which differ from t.

The subclass H of I which we have just defined contains most, but not all, of the particular iterative procedures which we discuss. All of the latter procedures, however, with one exception, are characterized by the fact that the corresponding functions $G_j(u, t)$ are linear in u. The only exception, the Germay-Ceressia method, is non-linear in a rather trivial way.

We shall verify that all the procedures discussed below have the strong property $P_I(f)$.

a) The Taylor expansion procedure. The vector function $f(u, t)$ is assumed to be analytic in u and t for $\| u - z_0 \| \leqq b$, $| t - t_0 | \leqq a$. Conditions $P_A(z_0)$ are therefore satisfied. We choose $y_0(t) = z_0$ and

$$G_{j+1}(u, t) = \sum_{r=0}^{j} \frac{(t-t_0)^r}{r!} \cdot \frac{d^r}{ds^r} f(y_j(s), s) \,|_{s=t_0}$$

The (vector) function $G_{j+1}(u, t)$ is thus independent of u. It is a functional of $y_j(t)$.

It is easily seen that the vector functions $y_j(t)$ have components which are partial sums of the corresponding Taylor expansions.

Conditions 2) and 3) of $P_I(f)$ are clearly satisfied. Conditions 1) (uniform boundedness) and 4') (uniform convergence) follow from Cauchy's theorem on the convergence of the infinite series of which components of $y_j(t)$ are partial sums.

b) Picard's Method of Successive Approximations. We set $H_j(u, y_{j-1}, y_{j-2}, \cdots, y_0, t) = f(y_{j-1}, t)$.

Strong property $P_I(f)$ is clearly satisfied.

c) The Lichtenstein method. We set $H_j(u, y_{j-1}, \cdots, y_0, t) = K_j(y_{j-1}, t)$, where $K_j(v, t)$ is a (vector) function converging to $f(v, t)$ uniformly in v and t as $j \to \infty$.

Strong property $P_1(f)$ is easily verified.

This procedure was proposed by Lichtenstein [3] in 1907 for single differential equations in order to study the dependence of solutions on initial conditions and parameters. Later (in 1947), it was re-introduced by Germay [4] for systems of differential equations without apparently any knowledge of Lichtenstein's paper.

d) The Kantorovitch procedure. The matrix $f_u(u, t)$ is assumed to exist and to be bounded within a domain of the u, t space including the point z_0, t_0.

We set

$$H_j(u, y_{j-1}, \cdots, y_0, t) = f_u(y_{j-1}(t), t)(u - y_{j-1}(t)) + f(y_{j-1}(t), t)$$

We easily verify strong property $P_1(f)$.

This method is a special case of Kantorovitch's [5] generalization of Newton's method to functions over Banach spaces.

e) The λ-procedure. (Expansion in terms of a parameter). In its generality, this procedure may be described as follows:

$f(u, t)$ is assumed to be analytic in u and continuous in t for $\| u - z_0 \| \leqq b$, $| t - t_0 | \leqq a$. $f(u, t)$ is considered as imbedded in a family of functions $h(u, t, \lambda)$, analytic in u and λ and continuous in t for $\| u - z_0 \| \leqq b$, $| t - t_0 | \leqq a$, $0 \leqq \lambda \leqq 1$, and $h(u, t, 1) = f(u, t)$.

Vector functions $q_r(t)$ are defined as follows: $q_0(t)$ is the solution of the system

$$\dot{q}_0 = h(q_0, t, 0), \quad q_0(t_0) = z_0$$

and q_r is the solution of

$$\text{(VI)} \qquad \dot{q}_r = (d^r/d\lambda^r)\, h(q_0(t) + \lambda q_1(t) + \cdots + (\lambda^r/r!)\, q_r(t), t, \lambda)\,|_{\lambda=0}$$

$$q_r(t_0) = 0, \qquad r \geqq 1.$$

(We use the symbol for total derivative to indicate differentiation with respect to λ in the first (vector) argument and in the third argument of $h(u, t, \lambda)$)

We then define the r-th iterate, $y_r(t)$, by

$$\text{(VII)} \qquad y_r(t) = q_0(t) + q_1(t) + \cdots + (1/r!)\, q_r(t).$$

Formally, this procedure consists in setting $z(t, \lambda)$ as the solution of $\dot{z} = h(z, t, \lambda)$, $z(t_0) = z_0$ and expanding components of $z(t, \lambda)$ as functions of λ into Taylor series about $\lambda = 0$, defining partial sums of these series to be components of the successive iterates. We have then: $q_r(t) = z_{\lambda^r}(t, 0)$.

This procedure is justified by the well known theorem (see, e.g., [6]) stating that $z(t, \lambda)$ is analytic in λ for $0 \leqq \lambda \leqq 1$.

In order to verify that the λ-procedure belongs to I and has strong property $P_I(f)$, we set:

$$H_j(u, y_{j-1}, \cdots, y_0, t) = \sum_{r=0}^{j-1} \frac{1}{r!} \frac{d^r}{d\lambda^r} h(p_{j-1}(t, \lambda), t, \lambda) \mid_{\lambda=0}$$

$$+ \frac{1}{j!} \frac{d^j}{d\lambda^j} h(p_{j-1}(t, \lambda) + \lambda^j(u - y_{j-1}(t)), t, \lambda) \mid_{\lambda=0},$$

where

$$p_j(t, \lambda) = y_0(t) + \lambda\{y_1(t) - y_0(t)\} + \cdots + \lambda^j\{y_j(t) - y_{j-1}(t)\}.$$

We observe that the system

$$\dot{y}_j = H_j(y_j, y_{j-1}(t) \cdots, y_0(t), t), \qquad y_j(t_0) = z_0$$

is equivalent to relations (VI) and (VII).

The convergence of $y_j(t)$ to the exact solution of system (A) is assured by the theorem just mentioned on the analyticity of $z(t, \lambda)$. We also have: $\lim (j \to \infty)\ p_j(t, \lambda) = z(t, \lambda)$. It can be easily verified that

$$H_j(u, y_{j-1}, y_{j-2}, \cdots, y_0, t) = \frac{\partial h(y_0, t, 0)}{\partial u}(u - y_j) + \sum_{r=0}^{j} \frac{1}{r!} \frac{d^r}{d\lambda^r} h(p_j(t, \lambda), t, \lambda) \mid_{\lambda=0}$$

Conditions 1), 2) and 3) of $P_I(f)$ follow easily. Condition 4') follows then from the second part (the converse) of Theorem I.

f) Liapunov's procedure. [7], [8]. For a description of this procedure we refer the reader to [8]. It follows easily from the results obtained there that strong property $P_I(f)$ is satisfied. Liapunov's procedure in many respects resembles the λ-procedure.

g) Germay's method (1). Germay proves in [9] the convergence of the following iterative method for single differential equations:

Let the equation be given in the form

$$\dot{z} = A(t)z + \phi(t, z), \qquad z(t_0) = z_0.$$

Then choose $y_0(t)$ such that $y_0(t_0) = z_0$ and let

$$\dot{y}_{j+1} = A(t)y_{j+1} + \phi(t, y_j(t)), \; y_{j+1}(t_0) = z_0.$$

Generalization to systems of *D.E.* is immediate if we allow $A(t)$ to be a matrix and z and $\phi(t, z)$ to be vectors. It should be noted, however, that the iterates y_{j+1} can be, in general, determined in a closed form only if the system is 1-dimensional.

Strong property $P_I(f)$ is verified if $A(t)$ is continuous and if $\phi(t, z)$ has property $P_A(z_0)$.

h) Germay's method (2). [10] This method applies only to systems of linear differential equations. Given the system

$$\dot{z} = A(t)z + f(t), \qquad z(t_0) = z_0,$$

we iterate as follows: $y_0(t)$ is continuous, $y_0(t_0) = z_0$ and

$$\dot{y}_{j+1} = K(y_{j+1}, y_j, t) + f(t), \; y_{j+1}(t_0) = z_0,$$

where $K(y_{j+1}, y_j, t)$ is obtained by replacing in the i-th component of $A(t)y_j$ the i-th component of y_j by the corresponding component of y_{j+1}.

The equations for y_{j+1} thus obtained may be integrated by quadratures.

Strong property $P_1(f)$ is obviously satisfied.

i) The Germay-Ceressia method. [11] This method is a generalization by Germay of an iterative method applied by Ceressia to systems of linear differential equations.

Given the system $\dot{z} = f(z, t)$, $z(t_0) = z_0$ written in the component notation

$$\dot{z}_i = f_i(z_1, \cdots, z_n, t), \qquad z_i(t_0) = z_{i0},$$

iterate as follows:

$$\dot{y}_{i,j+1} = f_i(y_{1,j+1}, y_{2,j+1}, \cdots, y_{i-1,j+1}, y_{i,j}, y_{i+1,j}, \cdots, y_{n,j}, t),$$

$$y_{i,j+1}(t_0) = z_{i0}$$

While the vector on the right hand side is, in general, not linear in the vector $(y_{1,j+1}, y_{2,j+1}, \cdots, y_{n,j+1})$, these equations may be integrated componentwise by successive quadratures.

Strong property $P_1(f)$ is clearly present.

4. Suggested Procedures. Of the methods described in the preceding paragraph, the one most generally valid is Picard's Method of Successive Approximations. On the other hand, as will be seen in §5, Kantorovitch's procedure is in general the one most rapidly convergent. Unfortunately, this last procedure is in most practical applications of very limited value. Indeed, the systems yielding the successive iterates are, except in trivial cases, linear with variable coefficients and thus not susceptible of being solved explicitly.

The A-procedure which we outline below strikes a certain compromise between Picard's and Kantorovitch's methods as to its applicability and convergence properties. It justifies, at the same time, certain estimates relating to the error analysis of approximate solutions of differential equations as applied, e.g., in [12].

Nothing was said so far about purely numerical methods of solving systems of differential equations. The iterative methods so far mentioned cannot be applied directly to numerical calculations because all substitutions must be made for all values of t and not only for discrete values. Kamke [13] mentions a combined iterative-numerical procedure but the latter does not converge to the exact solution of system (A).

It may be of interest to consider iterative numerical procedures which do converge to the exact solution of (A). We construct a family of such procedures, "the iterative k-th degree procedures" and "the generalized k-th degree procedures" which are shown to belong to the class I.

A k-th degree procedure is essentially a modification of Picard's method in which $f(y_j(t), t)$ is approximated by arcs of k-th degree parabolas and $y_{j+1}(t)$ is evaluated at a pointset K_{j+1} which becomes everywhere dense on the interval $(t_0, t_0 + T)$ as $j \to \infty$.

We may add that a k-th degree procedure and the A-procedure could be combined in a rather obvious manner.

1) *The A-procedure.* It is assumed that the matrix $(\partial/\partial u)\, f(u, t)$ exists for $u = z_0, t = t_0$. Let A be the constant matrix $(\partial/\partial u)\, f(z_0, t_0)$.

We set

$$G_j(u, t) = H_j(u, y_{j-1}, \cdots, y_0, t) = A(u - y_{j-1}) + f(y_{j-1}, t).$$

Strong property $P_I(f)$ is clearly present.

2) *The "iterative k-th degree procedure".* The vector function $f(u, t)$ is assumed to have continuous $(2k + 1) - st$ partial derivatives in the domain $\bar{D}_1$ in which $\| u - z_0 \| \leqq b$, $| t - t_0 | \leqq a$. We set Max $\| f(u, t) \|$ (in $\bar{D}_1$) $= M_f$ and we restrict ourselves to an interval $(t_0, t_0 + T_1)$ of the t-axis, where $T_1 \leqq$ Min $(a, b(M_f + \alpha)^{-1}, bM^{-1})$, α is some arbitrary positive number and M is an upper bound of $\| \dot{y}_0(t) \|$ for $| t - t_0 | \leqq a$.

The values of the 0-th approximation, $y_0(t)$, are tabulated at $s + 1$ discrete points in the interval $(t_0, t_0 + T_1)$, where s is sufficiently large depending on the magnitude of α and s is a multiple of k. We assume further, for the sake of simplicity, that these $s + 1$ points are equidistant, forming subintervals of length $h_0 = T_1/s$.

The values of the components of $f(y_0(t), t)$ are computed at the points $t_i = t_0 + ih_0$, $i = 0, 1, 2, \cdots, s$. The subintervals of length h_0 are then grouped into adjoining segments of length kh_0 and arcs of k-th degree parabolas are constructed, each arc passing through the $(k + 1)$ points $(t_i, f_\nu(y_0(t_i), t_i))$ corresponding to a given segment. (f_ν is the ν-th component of f.)

The vector function $G_1(u, t) = H_1(t)$ is then determined as having components consisting of the arcs of k-th degree parabolas just constructed.

The first iterate, $y_1(t)$, is obtained from $\dot{y}_1 = H_1(t)$, $y_1(t_0) = z_0$. This iterate $y_1(t)$ is then numerically evaluated at $2s + 1$ points, consisting of endpoints and midpoints of the previously considered intervals.

In general, the j-th iterate $y_j(t)$ is numerically known at $2^j s + 1$ equidistant points separated by the distance $h_j = 2^{-j}h_0$. The vector function $G_{j+1}(u, t) = H_{j+1}(t)$ has components consisting of arcs of k-th degree parabolas, the d-th arc passing through $(k + 1)$ points $(t_{ji}, f_\nu(y_j(t_{ji}), t_{ji}))$, $t_{ji} = t_0 + ih_j$, $(d - 1)k \leqq i \leqq dk$. The $(j + 1)$-st iterate, $y_{j+1}(t)$, is then obtained from $\dot{y}_{j+1} = H_{j+1}(t)$, $y_{j+1}(t_0) = z_0$ and evaluated at $2^{j+1}s + 1$ equidistant points from t_0 to $t_0 + T_1$.

We shall now prove

THEOREM II: If h_0 is chosen sufficiently small, then the "iterative k-th degree procedure" just defined belongs to the class I and satisfies strong property $P_I(f)$.

PROOF: We introduce the following notations:

Let $P_m(x, x_1, \cdots, x_m)$, $m = 0, 1, 2, \cdots, (2k+1)$, be polynomials in the scalar variables $x, x_1, \cdots, x_m$ obtained by replacing in the expansion of the vectors $(d^m/dt^m)f(v(t), t)$ the partial derivatives of the vector $f(u, t)$ by the maxima in $\bar{D}_1$ of their norms and replacing each component of $(d^r/dt^r)v(t)$ by x_r. We observe that P_m are non-decreasing functions of their arguments, so long as the latter are non-negative. Let, moreover, $b_1, b_2, \cdots, b_{k+1}$ be positive constants defined by:

$$b_1 = \text{Max } (P_0(b) + \alpha, \text{ Max } \| \dot{y}_0(t) \|) = \text{Max } (M_f + \alpha, M)$$

$$b_2 = \text{Max } (P_1(b, b_1) + \alpha, \text{ Max } \| \ddot{y}_0(t) \|)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_{k+1} = \text{Max } (P_k(b, b_1, \cdots, b_k) + \alpha, \text{ Max } \| y_0^{(k+1)}(t) \|).$$

We now consider the vector function

$$f(y_j(t), t) - H_{j+1}(t).$$

It is 0 for all points $t_{ji} = t_0 + ih_j$ and is $(2k+1)$ times differentiable inside each of the adjoining segments $(t_{j,(d-1)k}, t_{j,dk})$, $d = 1, 2, \cdots, T_1 k^{-1} h_j^{-1}$, of length kh_j.

It is well known([14]) that

$$\text{(VIII)} \qquad f(y_j(t), t) - H_{j+1}(t) = \frac{1}{(k+1)!} \frac{d^{k+1}}{dt^{k+1}} f(y_j(\xi), \xi) \cdot \phi_j(t),$$

where ξ and t belong to the same segment, $\phi_j(t) = (t - t_0')(t - t_1') \cdots (t - t_k')$, t_0' is the left endpoint of the segment and $t_r' = t_0' + rh_j$.

We also have ([15])

$$\frac{d^m}{dt^m} H_{j+1}(t) = H_{j+1}^{(m)}(t) = \frac{d^m}{dt^m} f(y_j(t), t)$$

$$- \sum_{\nu=0}^{m} \frac{m!}{\nu!(m+k+1-\nu)!} \frac{d^{m+k+1-\nu}}{dt^{m+k+1-\nu}} f(y_j(\xi_\nu), \xi_\nu) \frac{d^\nu}{dt^\nu} \phi_j(t), \; m \leqq k$$

$$H_{j+1}^{(m)}(t) = 0, \qquad m \geqq k+1,$$

where $t_0' < \xi_\nu < t_k'$. These last relations and the equation $\dot{y}_{j+1} = H_{j+1}(t)$ yield, after setting $Y_{j,r} = \text{Max } \| y_j^{(r)} \|$ $(t_0 \leqq t \leqq t_0 + T_1)$,

$$\text{(IX)} \qquad \begin{aligned} Y_{j+1,m+1} &\leqq P_m(Y_{j,0}, Y_{j,1}, \cdots, Y_{j,m}) \\ &+ \sum_{\nu=0}^{m} \frac{m!}{\nu!(m+k+1-\nu)!} P_{m+k+1-\nu}(Y_{j,0}, \cdots, Y_{j,k+1}, 0, \cdots, 0) \\ &\cdot \frac{d^\nu}{dt^\nu} \phi_j(t) \end{aligned}$$

We may easily verify that

$$\text{(X)} \qquad \left| \frac{d^\nu}{dt^\nu} \phi_j(t) \right| \leqq \frac{(k+1)!}{(k+1-\nu)!} k^{k+1-\nu} h_j^{k+1-\nu} < c_1 h_0^{k+1-\nu},$$

where c_1 is a constant, depending only on k.

We now require that h_0 be sufficiently small so that

$$\text{(XI)} \qquad c_1 \sum_{\nu=0}^{m} \frac{m!}{\nu!(m+k+1-\nu)!} P_{m+k+1-\nu}(b, \cdots, b_{k+1}, 0, \cdots, 0) h_0^{k+1-\nu} \leqq \alpha, \qquad m = 0, 1, \cdots, k$$

We may do so because the expressions on the left are polynomials in h_0 which vanish for $h_0 = 0$. There exists therefore a positive $h_0 = h_0(\alpha)$ such that the $(k+1)$ above inequalities are satisfied.

We shall prove now that $\| y_j(t) - z_0 \| \leqq b$ and $\| y_j^{(r)}(t) \| \leqq b_r$, $r \leqq k+1$, for all j.

This is true for $j = 0$ and we shall therefore make the inductive hypothesis that it is true for $j \leqq q$.

(IX), (X) and (XI) yield then:

$$Y_{q+1,m+1} \leqq P_m(b, b_1, \cdots, b_m) + \alpha \leqq b_{m+1}, \quad m = 0, 1, \cdots, k$$

Moreover,

$$\| y_{q+1}(t) - z_0 \| = \| \int_{t_0}^{t} \dot{y}_{q+1}(\tau) d\tau \| \leqq (t - t_0) Y_{q+1,1} \leqq T_1 b_1 \leqq b.$$

Having thus verified that $y_j(t)$ and their first $k+1$ derivatives remain uniformly bounded, we may conclude from (VIII) and (X) that

$$\text{(XII)} \qquad \| f(y_j(t), t) - H_{j+1}(t) \| \leqq c h_j^{k+1},$$

where c is a constant.

Thus condition (4′) of strong property $P_I(f)$ is verified. Conditions 2) and 3) are obviously satisfied and condition 1) holds with $M_\sigma = b_1$. This completes the proof of Theorem II.

An entirely analogous argument will also prove

THEOREM II′: If the iterative k-th degree procedure is generalized in such a manner that:

a) the j-th iterate $y_j(t)$ is numerically determined on a pointset K_j consisting of $s_j + 1$ points (not necessarily equidistant), $s_j \equiv 0 (\text{mod } k)$ and K_j becomes everywhere dense on the interval $(t_0, t_0 + T_1)$ as $j \to \infty$,

b) the vector function $H_{j+1}(t)$ has components consisting of arcs of k-th degree parabolas, the d-th arc passing through points $(t_i, f_\nu(y_j(t_i), t_i))$, where $t_i \in K_j$, $(d-1)k \leqq i \leqq dk$ and $t_\sigma < t_\rho$ if $\sigma < \rho$,

c) the largest interval determined by pointset K_0 is sufficiently small,

then the generalized k-th degree procedure belongs to I and has strong property $P_I(f)$.

5. Convergence properties of particular procedures belonging to I. Let us recall relation (V') of §2. This relation has been derived assuming strong property $P_I(f)$ to hold and assuming $t_0 = 0$, $z_0 = 0$. This relation will clearly remain valid, if we replace t by $t - t_0$ whenever $t_0 \neq 0$. ($m_0(s)$ is to be replaced then by $m_0(s + t_0)$ under the integral sign.)

The following conclusion is immediate:

LEMMA 3. If, for all j, $G_{j+1}(y_j(t), t) = f(y_j(t), t)$, then $\delta_j = 0$ for all j and

$$\| v_j(t) \| \leqq 2be^{K_G(t-t_0)}D^j(t-t_0)^j/j!, \tag{XIII}$$

as $m_0(s) = \| v_0(s)e^{-K_G s} \| \leqq 2be^{-K_G s} \leqq 2b$.

COROLLARY. In particular, if $f(u, t)$, $y_0(t)$ and $G_j(u, t)$ are analytic in all their arguments and $G_j(y_{j-1}(t), t) = f(y_{j-1}(t), t)$ for all j, then the j-th iterate $y_j(t)$ has at least the first $(j+1)$ terms of its Taylor expansion identical with the corresponding terms in the expansion of the exact solution $z(t)$. Indeed, $v_j(t) = z(t) - y_j(t)$ is then an analytic (vector) function for $t_0 \leqq t \leqq t_0 + T$ which vanishes at $t = t_0$. Hence $\| v_0(t) \| \leqq (b + M_1)(t - t_0)$, $t_0 \leqq t \leqq t_0 + T$, where $\| y_0(t) - z_0 \| \leqq M_1(t - t_0)$, and $m_0(t) \leqq (b + M_1)(t - t_0)e^{-K_G(t-t_0)}$. But

$$e^{K_G(t-t_0)} \int_{t_0}^{t} (t-s)^{j-1}(s-t_0)e^{-K_G(s-t_0)}\,ds$$

$$\leqq e^{K_G(t-t_0)} \int_{t_0}^{t} (s-t_0)(t-s)^{j-1}\,ds = e^{K_G(t-t_0)}(t-t_0)^{j+1}/j(j+1)$$

and thus

$$\| v_j(t) \| \leqq (b + M_1)e^{K_G \bar{t}}D^j\bar{t}^{j+1}/(j+1)! \leqq C\bar{t}^{j+1}/(j+1)!,$$

where C is a constant and $\bar{t} = t - t_0$. This last inequality implies that the expansion of $v_j(t)$ begins with the term in $\bar{t}^{j+1}$.

Relation (V') of §2 implies also

LEMMA 4. If strong property $P_1(f)$ holds and if the functions $G_j(u, t)$ are independent of u, then

$$\| v_j(t) \| \leqq \sum_{r=0}^{j-1} K_f^r \bar{t}^{r+1}\delta_{j-r-1}/(r+1)! + 2bK_f^j\bar{t}^j/j! \tag{XIV}$$

Indeed, in that case $K_G = 0$.

We shall now consider certain convergence properties characterizing the procedures described in §3 and §4, omitting discussion of the Taylor Series Expansion and Liapunov's procedure.

Picard's Method of Successive Approximations. Lemmas 3 and 4 apply and we have:

$$\| v_j(t) \| \leqq 2bK_f^j\bar{t}^j/j!$$

This relation also follows easily from the proof of Picard's Existence Theorem.

Furthermore, if $y_0(t)$ is chosen so as to satisfy a Lipschitz condition with some constant M_1, the argument of the Corollary to Lemma 3 remains valid and we have:

$$\| v_j(t) \| \leqq (b + M_1)K_f^j\bar{t}^{j+1}/(j+1)!$$

That this last relation cannot be much improved is shown by the following example:

$$\dot{z} = z + 1,\ z(0) = 0, \qquad y_0(t) = -M_1 t$$

We find easily

$$v_j(t) > (M_1 + 1)t^{j+1}/(j+1)!$$

The Lichtenstein Procedure. We may apply Lemma 4, choosing

$$\delta_j = \text{Max} \parallel f(v, t) - K_j(v, t) \parallel \qquad (v, t \in \bar{D})$$

The Kantorovitch Procedure. This procedure, like Newton's method, is characterized by the remarkable property that the error of any iterate is bounded, in general, by a constant times the square of the error of the preceding iterate. We have:

$$\parallel v_j(t) \parallel \leqq C\, C_1^{2^j-1} 2^{-(j+1)(j+2)/2} t^{2^j-1},$$

where C and C_1 are constants.

A similar relation was established by Kantorovitch in [4] for functional equations in Banach spaces. We shall include here a short proof relating to systems of differential equations in normal form for the sake of completeness.

We set $t_0 = 0$, $z_0 = 0$, which does not affect the generality of the argument. It is assumed that $f_{uu}(u, t)$ exists for $\parallel u \parallel \leqq b$ and $t \leqq T$. Furthermore, we assume $\parallel f_{uu}(u, t) \parallel \leqq 2\bar{K}$, $u \in \bar{D}$, where we define

$$\parallel f_{uu}(u, t) \parallel = \underset{p}{\text{Max}} \sum_{m=0,k=0}^{n} \left| f_{p_{u_m u_k}}(u, t) \right| .$$

We have

$$\dot{z} = f(z, t)$$

$$\dot{y}_{j+1} = f_u(y_j(t), t)(y_{j+1} - y_j) + f(y_j(t), t).$$

Subtracting, we obtain

$$\text{(XV)} \quad \dot{z} - \dot{y}_{j+1} = \dot{v}_{j+1} = f(z, t) - f(y_j(t), t) - f_u(y_j(t), t)(y_{j+1}(t) - y_j(t)).$$

But, by the Mean Value Theorem,

$$f(z, t) - f(y_j, t) = f_u(y_j, t)(z - y_j) + \tfrac{1}{2} \cdot f_{uu}(y_j + \theta v_j, t)(z - y_j)(z - y_j),$$

where $\theta = \theta(t)$, $0 \leqq \theta \leqq 1$, hence

$$\parallel f(z, t) - f(y_j, t) - f_u(y_j, t)(y_{j+1} - y_j) \parallel$$

$$= \parallel f_u(y_j, t)(z - y_{j+1}) + \tfrac{1}{2} \cdot f_{uu}(y_j + \theta v_j, t)(z - y_j)(z - y_j) \parallel$$

$$\leqq K_f \parallel v_{j+1} \parallel + \bar{K} \parallel v_j \parallel^2.$$

Referring to equation (XV), we conclude that

$$\parallel v_{j+1} \parallel \leqq K_f \int_0^t \parallel v_{j+1}(s) \parallel ds + \bar{K} \int_0^t \parallel v_j(s) \parallel^2 ds.$$

Applying Lemma 2, we find

$$\| v_{j+1} \| \leqq \bar{K} e^{K_f t} \int_0^t e^{-K_f s} \| v_j(s) \|^2 \, ds \leqq M_2 \int_0^t \| v_j(s) \|^2 \, ds, \text{ where } M_2 = \bar{K} e^{K_f T}$$

Now $\| v_0(t) \| \leqq L$, where L is a constant, hence

$$\| v_1 \| \leqq M_2 L^2 t, \qquad \| v_2 \| \leqq M_2^{1+2} L^{2^2} t^3/3.$$

$$\| v_j \| \leqq M_2^{1+2+\cdots 2^{j-1}} L^{2^j} t^{2^j-1}/3 \cdot 7 \cdot 15 \cdots (2^j - 1).$$

We observe that

$$3 \cdot 7 \cdot 15 \cdots (2^j - 1) = \prod_{r=1}^{j} (2^r - 1)$$

$$= \prod_{r=1}^{j} 2^r \prod_{r=1}^{j} (1 - 2^{-r}) \geqq 2^{j(j+1)/2} \prod_{r=1}^{\infty} (1 - 2^{-r}),$$

hence $\| v_j \| \leqq C (M_2 L)^{2^j-1} 2^{-j(j+1)/2} t^{2^j-1}$, where

$$C = L \prod_{r=1}^{\infty} (1 - 2^{-r})^{-1}.$$

The λ-procedure. In the general case (see relation VII of §3)

$$y_j(t) = \sum_{r=0}^{j} q_r(t)/r!, \qquad \text{where } q_r(t) = \frac{d^r}{d\lambda^r} z(t, 0).$$

By "appropriately" choosing $h(u, t, \lambda)$ we may obtain very fast or very slow convergence. Some more detailed information about the type of convergence may be obtained in case $h(u, t, \lambda)$ is linear in λ. We have then:

THEOREM III: If $f(u, t) = \varphi(u, t) + \psi(u, t)$ and $h(u, t, \lambda) = \varphi(u, t) + \lambda\psi(u, t)$, then the λ-procedure is characterized by relation

$$\| q_r(t)/r! \| \leqq A_r \bar{t}^r, \tag{XVI}$$

where A_r is independent of t.

PROOF: $h(u, t, \lambda) = \varphi(u, t) + \lambda\psi(u, t)$ implies

$$\frac{d^r}{d\lambda^r} h(z(t, 0), t, 0) = \frac{d^r}{d\lambda^r} \varphi(z(t, 0), t) + r \frac{d^{r-1}}{d\lambda^{r-1}} \psi(z(t, 0), t).$$

Moreover, $\| q_0(t) \|$ is bounded, hence relation (XVI) is true for $r = 0$.

We shall now assume, for purposes of induction, that

$$\| q_k(t)/k! \| \leqq A_k \bar{t}^k \qquad \text{for } k < r.$$

We observe that $(d^r/d\lambda^r)\varphi(z(t, 0), t)$ is a polynomial in the derivatives with respect to λ of $z(t, \lambda)$ at $\lambda = 0$ with bounded coefficients (which are partial derivatives of $f(u, t)$ evaluated inside $\bar{D}_1 : \| u - z_0 \| \leqq b, | t - t_0 | \leqq a$). This polynomial is of the form

$$\sum K^{\beta_1, \cdots, \beta_i}_{\alpha_1, \cdots, \alpha_i}(t) z^{\beta_1}_{\lambda^{\alpha_1}} \cdots z^{\beta_i}_{\lambda^{\alpha_i}},$$

where $z_{\lambda^\alpha} = (d^\alpha/d\lambda^\alpha)z(t, 0)$ and $\sum_\alpha \alpha_\alpha\beta_\alpha = r$. A similar remark applies to $(d^{r-1}/d\lambda^{r-1})\psi(z(t, 0), t)$. Our inductive hypothesis implies then

$$\left\| \frac{d^r}{d\lambda^r} h(z(t, 0), t, 0) \right\| \leqq \left\| \frac{\partial\varphi(y_0(t), t)}{\partial u} \right\| \cdot \| q_r(t) \| + B_r\bar{t}^r + C_{r-1}\bar{t}^{r-1}.$$

Combining this result with relation (VI) of §3, we find

$$\| q_r(t) \| \leqq \int_{t_0}^{t} \left\| \frac{d^r}{d\lambda^r} h(z(s, 0), s, 0) \right\| ds \leqq E \int_{t_0}^{t} \| q_r(s) \| ds + B_r(t - t_0)^{r+1}/(r + 1) + C_{r-1}(t - t_0)^r/r,$$

where E is the maximum of

$$\| (\partial/\partial u)\varphi(y_0(t), t) \| \qquad (t_0 \leqq t \leqq t_0 + T)$$

We may apply Lemma 2, which yields

$$\| q_r(t) \| \leqq B_r(t - t_0)^{r+1}/(r + 1) + C_{r-1}(t - t_0)^r/r + Ee^{E(t-t_0)} \int_0^{\bar{t}} e^{-Es} \cdot \{B_r s^{r+1}/(r + 1) + C_{r-1} s^r/r\} ds$$

For $t - t_0 \leqq T$, the expression on the right is bounded by $A_r\bar{t}^r$, where A_r is some constant. This completes our inductive proof of relation (XVI).

Relation (XVI) also shows that, if $h(u, t, \lambda)$ is linear in λ and analytic in all its arguments, then the j-th iterate $y_j(t)$ has at least $(j + 1)$ terms of its Taylor expansion identical with the corresponding terms of the exact solution $z(t)$.

Germay's method (1). Lemma 3 is applicable. Thus, if $A(t)$ and $\phi(z, t)$ are analytic, each iteration will add one more correct term to the Taylor expansion of the solution. That the method will not in general yield better results is shown by the following example:

$$\dot{z} = z/(1 - t) + z^2, \qquad z(0) = 1.$$

The expansion of the exact solution is

$$z = 1 + 2t + 7/2\, t^2 + 35/6\, t^3 + \cdots$$

Let $y_0(t) = 1 + 2t$. Then

$$y_1(t) = 1 + 2t + 7/2\, t^2 + 7/2\, t^3 + \cdots$$

Germay's method (2). This procedure has somewhat similar convergence properties as the previous one. Lemma 3 is applicable. The following example will show that, in case of analyticity, no more than one correct term is added with each iteration to the Taylor series of the solution.

$$\dot{x} = -1/3\, x - y, \; x(0) = 3$$

$$\dot{y} = -2x + 2y, \; y(0) = 2$$

The expansion of the exact solution is

$$x = 3 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{2}t^3 + \tfrac{1}{8}t^4 + \cdots$$
$$y = 2 - 2t + t^2 - \tfrac{1}{3}t^3 + \tfrac{1}{12}t^4 + \cdots$$

We choose

$$x_0(t) = 3 - 3t$$
$$y_0(t) = 2 - 2t$$

One application of this procedure yields:

$$x_1(t) = 3 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{72}t^4 + \cdots$$
$$y_1(t) = 2 - 2t + t^2 + \tfrac{3}{3}t^3 + \tfrac{1}{3}t^4 + \cdots$$

The Germay-Ceressia method. Lemma 3 is applicable. In case of analyticity, each iteration yields in general only one additional correct term in the Taylor expansion of the solution.

We use as illustration the previous example, choosing the same $x_0(t)$ and $y_0(t)$. The first iterate is then

$$x_1(t) = 3 - 3t + \tfrac{3}{2}t^2 - t^3 + \cdots$$
$$y_1(t) = 2 - 2t + t^2 + t^3 + \cdots$$

Suggested Procedures. The A-procedure. We shall prove

THEOREM IV: If $f_u(u, t)$ exists and satisfies a Lipschitz condition in the vector variable u and in t in the domain $\bar{D}$: $\| u - z_0 \| \leqq b$, $| t - t_0 | \leqq T$, then the A-procedure is characterized by the relation

$$\| v_j(t) \| \leqq LP^{j-1}t^{2j-1}/(j-1)!,$$

where L and P are constants.

PROOF: We assume $t_0 = 0$, $z_0 = 0$

We have

$$\dot{z} = f(z, t)$$
$$\dot{y}_{j+1} = A(y_{j+1} - y_j) + f(y_j, t)$$

hence

$$\dot{v}_{j+1} = Av_{j+1} + f(z, t) - f(y_j, t) - Av_j \tag{XVII}$$

Now

$$f(z, t) = f(y_j, t) + f_u(y_j + \theta v_j, t)v_j,$$

where $\theta = \theta(t)$ and $0 \leqq \theta \leqq 1$. Thus

$$f(z, t) - f(y_j, t) - Av_j = \{f_u(y_j + \theta v_j, t) - f_u(0, 0)\}v_j$$

We observe that, for $j \geqq 1$,

$$\| y_j \| \leqq C_1 t,$$

where C_1 is a constant. Moreover, for $j \geqq 1$,

$$\| v_j \| \leqq C_2 t,$$

as implied by Lemma 3. Thus

$$\| y_j + \theta v_j \| \leqq C_3 t, \qquad j \geqq 1,$$

and, because of the Lipschitz condition,

$$\| f_u(y_j + \theta v_j, t) - f_u(0, 0) \| \leqq C_4 t.$$

Referring to equation (XVII), we conclude that

$$\| v_{j+1} \| \leqq \| A \| \int_0^t \| v_{j+1}(s) \| \, ds + C_4 \int_0^t s \| v_j(s) \| \, ds, \quad j \geqq 1.$$

We apply Lemma 2 and find:

$$\| v_{j+1} \| \leqq C_4 \int_0^t s \| v_j(s) \| \, ds + C_4 M e^{MT} \int_0^t e^{-Ms} \, ds \int_0^s \tau \| v_j(\tau) \| \, d\tau, \quad j \geqq 1,$$

where $M = \| A \|$.

An elementary calculation shows that the right hand side of the inequality is equal to

$$C_4 e^{Mt} \int_0^t s e^{-Ms} \| v_j(s) \| \, ds \leqq C_4 e^{MT} \int_0^t s \| v_j(s) \| \, ds$$

and thus

$$\| v_{j+1}(t) \| \leqq R \int_0^t s \| v_j(s) \| \, ds, \quad R = C_4 e^{MT}.$$

Now, by relation (V′) of §2,

$$\| v_1(t) \| \leqq Lt, \quad \text{where} \quad L = (K_f + K_g) e^{K_g T} V_0 \quad \text{and} \quad V_0 \geqq \| v_0(t) \|,$$

hence

$$\| v_2(t) \| \leqq LRt^3/3,$$
$$\| v_3(t) \| \leqq LR^2 t^5/3.5$$
$$\cdots\cdots\cdots$$
$$\| v_j(t) \| \leqq LR^{j-1} t^{2j-1}/3.5 \cdots (2j-1) \leqq LP^{j-1} t^{2j-1}/(j-1)!,$$

where $P = \frac{1}{2}R$.

The iterative k-th degree procedure. We may clearly apply Lemma 4. Moreover, relation (XII) of Theorem II (§4) implies:

$$\delta_j \leqq c h_j^{k+1} = c h_0^{k+1} (2^{-k-1})^j = N(2^{-k-1})^j,$$

where N is a constant. Thus

$$\| v_j(t) \| \leqq \sum_{r=0}^{j-1} N(2^{-k-1})^{j-r-1} K_f^r t^{r+1}/(r+1)! + 2bK_f^j t^j/j!$$
$$= N(2^{-k-1})^j \sum_{r=0}^{j-1} K_f^r (2^{k+1} t)^{r+1}/(r+1)! + 2bK_f^j t^j/j!$$
$$\leqq N(2^{-k-1})^j \{\exp(2^{k+1} K_f t) - 1\} K_f^{-1} + 2bK_f^j t^j/j!$$

Now $K_f^{-1}\{\exp(2^{k+1}K_f\bar{t}) - 1\}$ is bounded for $\bar{t} \leqq T$ and for all K_f. (In the trivial case $K_f = 0$, we must clearly replace $K_f^{-1}\{\exp(2^{k+1}K_f\bar{t}) - 1\}$ by $2^{k+1}\bar{t}$).

We may therefore state

THEOREM V: The iterative k-th degree procedure is characterized by:

$$\| v_j(t) \| \leqq C(2^{-k-1})^j + 2bK_f^j\bar{t}^j/j!, \tag{XVIII}$$

where C is a constant.

We may derive in an analogous manner

THEOREM V′: If the pointset K_j (of Theorem II′) consists of equidistant points separated by intervals of length $h_j = h_0p^{-j}$, where p is an integer $\geqq 2$, then the corresponding generalized k-th degree procedure is characterized by:

$$\| v_j(t) \| \leqq C(p^{-k-1})^j + 2bK_f^j\bar{t}^j/j!,$$

where C is a constant.

Appendix. On Iterative Procedures for Solving Integral and Functional Equations. The class of iterative procedures for solving differential equations which we have defined in §2 may be generalized to apply to a wider class of equations. These procedures are, as a matter of fact, applicable to equations which are susceptible of solution by a Picard-type iteration. One of these procedures, due to Kantorovitch, was actually introduced in a form applicable to equations over a Banach space.

Let us now consider a function space and some (non-linear) operator F on the points of the space. It is obvious that the equation

$$z = Fz \tag{B}$$

may be uniquely solved by successive substitutions (Picard-type iteration) provided

$$\| Fx - F\bar{x} \| \leqq K \| x - \bar{x} \| \quad \text{and} \quad K < 1.$$

We may now define points y_j satisfying the equations

$$y_j = G_jy_j, \tag{Ap I}$$

where

1) $\| G_jx - G_j\bar{x} \| \leqq K_a \| x - \bar{x} \|$ for all j and $2K_a + K < 1$

2) $\| G_{j+1}y_j - Fy_j \| \to 0$ as $j \to \infty$.

It can be easily proven that

$$\| y_j - z \| \to 0 \qquad \text{as } j \to \infty.$$

Thus the equations (Ap I) define a class of iterative procedures, analogous to I and converging to the solution of equation (B).

As an example we may consider the Fredholm-type integral equation

$$z(t) = \int_a^b F(t, s, z(s))\,ds + h(t) \tag{C}$$

which may, under certain conditions, be solved by a Picard-type iteration [16]. The equations

$$\text{(Ap II)} \qquad y_j(t) = \int_a^b G_j(t, s, y_j(s))\, ds + h_j(t)$$

will define a sequence of functions $y_j(t)$ converging to $z(t)$ provided conditions analogous to $P_1(f)$ are satisfied and the Lipschitz constant of $G_j(t, s, u)$ with respect to u is sufficiently small.

This last condition is no more required if we consider Volterra-type equations obtained by replacing b by t in the upper limits of the integrals in (C) and (Ap II).

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APPLICATIONS OF THE THEORY OF SYSTEMS OF DIFFERENTIAL EQUATIONS TO MULTIPLE NON-UNIFORM TRANSMISSION LINES

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1. Introduction. The purpose of this paper is to apply recent developments in the theory of systems of differential equations, particularly certain results of Reid [3] and Sternberg [5], to the theory of multiple non-uniform transmission lines in the steady state. The case of multiple uniform transmission lines in the steady state has recently been investigated by Rice [4].

The equations of the multiple non-uniform transmission line considered are postulated in §2 in the form of two ordinary linear vector differential equations of the first order. In §3 two Riccati type matrix differential equations associated with the multiple line are derived and continuous symmetric matrix solutions of these equations are identified as matrix analogues of the usual impedance and admittance functions of a simple line. It is then shown that if a continuous symmetric matrix solution of either of these equations is known then the general solution for the complex currents and voltages along the multiple line may be determined by solving a single linear matrix differential equation of the first order. In §4 several theorems are given concerning the non-existence of multiple voltage nodes on a lossless multiple line of either finite or infinite length. Among other results obtained, it is found that a necessary and sufficient condition for the non-existence of more than one multiple voltage node on a finite lossless multiple line, or for large x on an infinite lossless multiple line, is that in the former case there exist along the finite multiple line a pure imaginary continuous symmetric matrix solution of the Riccati type matrix differential equation for the matrix admittance of the multiple line derived in §3 and in the latter case that there exist such a matrix solution for large x. Similar theorems are noted to hold for currents.

Vector and matrix notation is used throughout. In particular, capital italic letters are used to denote $n \times n$ square matrices while the prime is used to denote the derivative and the asterisk to denote the transpose of a vector or matrix. We call a matrix solution of a matrix differential equation continuous when each of its elements is a continuous function of x in the usual sense.

2. Transmission Line Equations. We consider a multiple non-uniform transmission line consisting of n wires with ground, or additional single wire return, extending from $x = a$ to $x = b$, $0 \leqq a < b \leqq \infty$, whose differential equations in the steady state can be written in the vector form

$$u' = -Y(x)v, \qquad v' = -Z(x)u, \tag{2.1}$$

where u and v are respectively the unknown complex current and voltage column vectors $u(x) \equiv (u_p(x))$, $v(x) \equiv (v_p(x))$, $(p = 1, 2, \cdots, n)$ with the actual instantaneous currents and voltages being the components of the real column vectors

$$i(x, t) \equiv \mathfrak{Re}[u(x)e^{i\omega t}], \qquad e(x, t) \equiv \mathfrak{Re}[v(x)e^{i\omega t}], \tag{2.2}$$

and where $Y(x)$ and $Z(x)$ are respectively the given square *specific shunt admittance* and *specific series impedance* matrices

$$Y(x) \equiv \| G_{pq}(x) + j\omega C_{pq}(x) \|, \qquad Z(x) \equiv \| R_{pq}(x) + j\omega L_{pq}(x) \|, \tag{2.3}$$
$$(p, q = 1, 2, \cdots, n).$$

In (2.1) and (2.3) we suppose that $G_{pq}(x) \equiv G_{qp}(x)$, $C_{pq}(x) \equiv C_{qp}(x)$, $R_{pq}(x) \equiv R_{qp}(x)$ and $L_{pq}(x) \equiv L_{qp}(x)$ or, in other words, that the matrices $Y(x)$ and $Z(x)$ are symmetric. Moreover, we suppose that each element of each of these matrices is continuous along the multiple line. By a *solution* $u = u(x)$, $v = v(x)$ of the equations (2.1) we will understand a pair of vectors as above whose elements are complex valued continuous functions each with a continuous first derivative and which satisfy the equations (2.1) identically along the multiple line. Finally, if $n = 1$ we call the line simple.

In the special case in which $Y(x)$ and $Z(x)$ are constant matrices, multiple transmission lines in the steady state having equations of the form (2.1) have been considered by Rice [4]. Several of the general results obtained below for multiple non-uniform transmission lines in the steady state will be found to have a special significance, or to reduce to results obtained by Rice [4], in the case of multiple uniform transmission lines.

3. Admittance and Impedance Equations. For any pair of solutions u, v and u^0, v^0 of (2.1) the symmetry of the matrices $Y(x)$ and $Z(x)$ implies that $u^*v^0 - v^*u^0 \equiv$ constant. If this constant is zero then the two solutions u, v and u^0, v^0 are said to be *conjugate* to each other. If $U = U(x)$, $V = V(x)$ are $n \times n$ matrices whose respective columns taken in pairs are mutually conjugate solutions of (2.1) then $U(x)$ and $V(x)$ are said to be *matrices of conjugate solutions* of (2.1). Such matrices thus satisfy the condition

$$U^*V - V^*U \equiv 0 \tag{3.1}$$

or, in other words, the matrix $U^*(x)V(x)$ is symmetric along the multiple line. With the aid of fundamental existence theorems for systems of differential equations we note in passing that for each point x_0 on the multiple line there always exist matrices $U_1(x)$, $V_1(x)$ and $U_2(x)$, $V_2(x)$ of conjugate solutions of (2.1) uniquely determined by suitable boundary conditions at the point x_0 in terms of which the general solution $u = u(x)$, $v = v(x)$ of (2.1) can always be expressed linearly.

Suppose now $U(x)$ and $V(x)$ are matrices of conjugate solutions of (2.1) with $V(x)$ non-singular along the multiple line. Then defining an $n \times n$ matrix $H(x) \equiv U(x)V^{-1}(x)$ one finds with the aid of (2.1) and (3.1) that $H = H(x)$ is a continuous symmetric matrix solution of the Riccati type matrix differential equation

$$H' - HZ(x)H + Y(x) = 0. \tag{3.2}$$

Similarly, if $U(x)$ is non-singular along the multiple line then the $n \times n$ matrix $K(x) \equiv V(x)U^{-1}(x)$ is a continuous symmetric matrix solution $K = K(x)$ of the Riccati type matrix differential equation

$$K' - KY(x)K + Z(x) = 0. \tag{3.3}$$

In case $U(x)$ and $V(x)$ are both non-singular then for $H(x) \equiv U(x)V^{-1}(x)$ and $K(x) \equiv V(x)U^{-1}(x)$ we have that $H(x)$ and $K(x)$ so defined are both non-singular and that $H^{-1}(x) \equiv K(x)$. In any case, regardless of the manner of definition, whenever an $n \times n$ continuous symmetric matrix solution $H = H(x)$ of (3.2) or $K = K(x)$ of (3.3) exists then the former may be interpreted as a *matrix analogue* for the multiple line of the usual *admittance function* of a simple line and the latter may be interpreted as a *matrix analogue* for the multiple line of the usual *impedance function* of a simple line, for in such a case we have the following important result.

Suppose that $H = H(x)$ is a given $n \times n$ continuous symmetric matrix solution of the Riccati type matrix differential equation (3.2) *along the multiple line. If now the matrix differential equation*

$$V_1' = -Z(x)H(x)V_1 \tag{3.4}$$

is solved for an $n \times n$ continuous non-singular matrix solution $V_1 = V_1(x)$ satisfying $V_1(a) = I$ and if $n \times n$ matrices $U_1(x)$, $U_2(x)$ $V_2(x)$ are defined as

$$\begin{aligned} U_1(x) &\equiv H(x)V_1(x), \\ U_2(x) &\equiv -U_1(x)\int_a^x V_1^{-1}(s)Z(s)V_1^{*-1}(s)\,ds + V_1^{*-1}(x), \\ V_2(x) &\equiv -V_1(x)\int_a^x V_1^{-1}(s)Z(s)V_1^{*-1}(s)\,ds, \end{aligned} \tag{3.5}$$

then the general solution of (2.1) *for the complex currents and voltages along the multiple line is give by*

$$u(x) = U_1(x)c + U_2(x)\,d, \qquad v(x) = V_1(x)c + V_2(x)\,d, \tag{3.6}$$

where c and d are the constant vectors $c = v(a)$ and $d = u(a) - H(a)v(a)$. Conversely, suppose that $K = K(x)$ is a given $n \times n$ continuous symmetric matrix solution of the Riccati type matrix differential equation (3.3) *along the multiple line. If now the matrix differential equation*

$$U_3' = -Y(x)K(x)U_3 \tag{3.7}$$

is solved for an $n \times n$ continuous non-singular matrix solution $U_3 = U_3(x)$ satisfying $U_3(a) = I$ and if $n \times n$ matrices $V_3(x)$, $U_4(x)$ and $V_4(x)$ are defined as

$$\begin{aligned} V_3(x) &\equiv K(x)U_3(x), \\ U_4(x) &\equiv -U_3(x)\int_a^x U_3^{-1}(s)Y(s)U_3^{*-1}(s)\,ds, \\ V_4(x) &\equiv -V_3(x)\int_a^x U_3^{-1}(s)Y(s)U_3^{*-1}(s)\,ds + U_3^{*-1}(x), \end{aligned} \tag{3.8}$$

then the general solution of (2.1) *for the complex currents and voltages along the multiple line is given by*

$$u(x) = U_3(x)c^0 + U_4(x)\,d^0, \qquad v(x) = V_3(x)c^0 + V_4(x)\,d^0, \tag{3.9}$$

where c^0 and d^0 are the constant vectors $c^0 = u(a)$ and $d^0 = v(a) - K(a)u(a)$. Finally, if both (3.6) *and* (3.9) *hold then the two solutions are equivalent.*

To establish the preceding result one may begin by applying fundamental existence theorems for systems of differential equations to show for a given matrix $H = H(x)$ as above satisfying (3.2) that a matrix solution $V_1 = V_1(x)$ of (3.4) of the type described exists, and then proceed as in the proof of Theorem 3.1 of Reid [3; pp. 242–243] to show that for $U_1(x) \equiv H(x)V_1(x)$ the equations (3.2) and (3.4), together with the symmetry of the matrix $H(x) \equiv U_1(x)V_1^{-1}(x)$, imply that $U_1(x)$ and $V_1(x)$ are matrices of conjugate solutions of (2.1). The proof of the first assertion is then readily completed by showing by substitution that the matrices $U_2(x)$, $V_2(x)$ defined in (3.5) are also matrices of conjugate solutions of (2.1) and by noting from the fact that the $2n \times 2n$ matrix

$$\left\| \begin{matrix} U_1(x) & U_2(x) \\ V_1(x) & V_2(x) \end{matrix} \right\| \tag{3.10}$$

is non-singular at $x = a$ that the columns of the matrices $U_1(x)$, $V_1(x)$, $U_2(x)$, $V_2(x)$ form a fundamental set of solutions of (2.1). Finally, the second assertion follows from the first by a notational transformation while the third assertion is an immediate consequence of the same fundamental existence theorems mentioned earlier.

Although the Riccati type matrix differential equations (3.2) and (3.3), and the linear matrix differential equations (3.4) and (3.7), will not in general be solvable in terms of elementary functions, nevertheless, there is available some knowledge concerning properties of solutions of matrix differential equations of the former type in Reid [3] and of both general types in a number of places; e.g., see Bellman's survey [1; pp. 4–88]. In the particular case of a multiple uniform line, however, the preceding result under suitable conditions leads to explicit formulas for the complex currents and voltages along the multiple line. Thus noting for constant symmetric matrix solutions $H = H_0$ of (3.2) or $K = K_0$ of (3.3) that those equations reduce to the algebraic matrix equations

$$H_0 Z(x) H_0 = Y(x), \tag{3.11}$$

$$K_0 Y(x) K_0 = Z(x), \tag{3.12}$$

we find in the case of a multiple uniform line, i.e., when $Y(x)$ and $Z(x)$ are constant matrices Y and Z, the following special results.

If for a finite multiple uniform line a constant symmetric non-singular matrix H_0 satisfying (3.11) *or K_0 satisfying* (3.12) *is known then in the first case the complex currents and voltages along the finite multiple uniform line are given by*

$$\begin{aligned} u(x) &= -H_0 \sinh [(x-a)ZH_0]v(a) + \cosh [(x-a)H_0Z]u(a), \\ v(x) &= \cosh [(x-a)ZH_0]v(a) - H_0^{-1} \sinh [(x-a)H_0Z]u(a), \end{aligned} \tag{3.13}$$

and in the second case by

$$\begin{aligned} u(x) &= \cosh [(x-a)YK_0]u(a) - K_0^{-1} \sinh [(x-a)K_0Y]v(a), \\ v(x) &= -K_0 \sinh [(x-a)YK_0]u(a) + \cosh [(x-a)K_0Y]v(a). \end{aligned} \tag{3.14}$$

If for an infinite multiple uniform line a constant symmetric non-singular matrix H_0 satisfying (3.11) *or K_0 satisfying* (3.12) *is known while the corresponding matrix limit*

$$\lim_{x\to\infty} e^{-(x-a)ZH_0} = 0, \text{ or } \lim_{x\to\infty} e^{-(x-a)YK_0} = 0, \tag{3.15}$$

exists then in the first case the complex currents and voltages along the infinite multiple uniform line are given by

$$u(x) = H_0 v(x), \qquad v(x) = e^{-(x-a)ZH_0}\, v(a), \tag{3.16}$$

and in the second case by

$$u(x) = e^{-(x-a)YK_0}\, u(a), \qquad v(x) = K_0 u(x). \tag{3.17}$$

Solutions for the complex currents and voltages along finite and infinite multiple uniform lines essentially equivalent to those given by (3.13), (3.14), (3.16) and (3.17) have been given by Rice [4; pp. 134–139]. To establish our solutions above one has only to apply the general method given earlier to the special cases considered. Thus, for example, to obtain (3.13) and (3.16) under the stated hypothesis note for these cases with $Z(x)$ and $H(x)$ constant matrices Z and H_0 as described, that the matrices $U_1(x)$, $V_1(x)$, $U_2(x)$, $V_2(x)$ defined by (3.4) and (3.5) are respectively the matrices

$$\begin{aligned} U_1(x) &\equiv H_0 e^{-(x-a)ZH_0}, & V_1(x) &\equiv e^{-(x-a)ZH_0}, \\ U_2(x) &\equiv \cosh\,[(x-a)H_0 Z], & V_2(x) &\equiv -H_0^{-1}\sinh\,[(x-a)H_0 Z], \end{aligned} \tag{3.18}$$

and then rearrange the general solution (3.6) suitably to obtain (3.13) and put $d = 0$ in (3.6) to obtain (3.16). We note in passing that the hypotheses (3.15) are introduced simply to conform with the physically plausible assumption that the currents and voltages along an infinite multiple line tend to zero with increasing x; the same assumption toegther with the hypotheses (3.15) then requires the above choice of the constant vector d in (3.6).

Following the terminology of Rice [4; p. 134] one may interpret the constant symmetric matrices H_0 and K_0 respectively as the *characteristic matrix admittance* and *characteristic matrix impedance* for the multiple uniform line while the matrices $\Gamma_1 = ZH_0$ and $\Gamma_2 = YK_0$ may be considered as *matrix analogues* of the usual *propagation constant* for a simple line. With regard to the latter we note in passing that in general for the multiple uniform line we have

$$\Gamma_1^2 = ZY = \Gamma_2^{*2} \tag{3.19}$$

while in case H_0 and K_0 are both non-singular and $H_0^{-1} = K_0$ then $\Gamma_1 = \Gamma_2^*$.

The Riccati type matrix differential equations (3.2) and (3.3) associated with the multiple line in the general non-uniform case also have another significance. Thus in §4 below a fundamental relation will be noted between the existence of continuous symmetric solutions $H = H(x)$ of (3.2) and the non-existence of more than one multiple voltage node on a finite lossless multiple line or for large x on an infinite lossless multiple line.

4. Non-Existence of Multiple Voltage or Current Nodes. Throughout the balance of the paper we suppose that the multiple non-uniform transmission line is *lossless*, i.e., we suppose in (2.1) and (2.3) that $G_{pq}(x) \equiv R_{pq}(x) \equiv 0$, $(p, q = 1, 2, \cdots, n)$ along the multiple line and write $C(x) \equiv \| C_{pq}(x) \|$, $L(x) \equiv \| L_{pq}(x) \|$, $(p, q = 1, 2, \cdots, n)$ for the indicated *specific shunt capacity* and *specific series inductance* matrices. Moreover, we suppose that each of the matrices $C(x)$ and $L(x)$ is *positive definite* along the multiple line and that the elements of these matrices are not only continuous functions of x but also have continuous first derivatives. The former of these hypotheses means specifically, that the quadratic forms $\pi^* C(x)\pi$ and $\pi^* L(x)\pi$ are positive functions of x for all constant column vectors $\pi \equiv (\pi_p)$, $(p = 1, 2, \cdots, n)$ with real elements not all zero and corresponds in the case of a lossless simple line to the assumption that the specific shunt capacity and specific series inductance of the line are positive functions. In particular, we have then that the determinants of the matrices $C(x)$ and $L(x)$ are different from zero along the multiple line and, thus, that these matrices are non-singular with inverse matrices $C^{-1}(x)$ and $L^{-1}(x)$ respectively. Hence, the equations (2.1) in the present case for the lossless multiple line are seen readily to be equivalent to either of the linear second order vector differential equations

$$[C^{-1}(x)u']' + \omega^2 L(x)u = 0, \tag{4.1}$$

$$[L^{-1}(x)v']' + \omega^2 C(x)v = 0, \tag{4.2}$$

under the transformations $v \equiv (j/\omega)C^{-1}(x)u'$ and $u \equiv (j/\omega)L^{-1}(x)v'$ respectively where by a solution $v = v(x)$ of (4.2) for example, we understand a vector $v(x) \equiv (v_p(x))$, $(p = 1, 2, \cdots, n)$ whose elements are complex valued continuous functions each with a continuous first derivative, which is such that the elements of the vector $L^{-1}(x)v'$ are also continuous with continuous first derivatives and which satisfies (4.2) identically along the lossless multiple line.

Since the two differential equations (4.1) and (4.2) are of the same form, and have coefficient matrices satisfying the same hypotheses, we will henceforth consider only the *voltage equation* (4.2); entirely similar results will be seen to hold for the *current equation* (4.1). To begin with, from the assumed properties of the matrix $L(x)$ it follows that the inverse matrix $L^{-1}(x)$ is symmetric and positive definite along the lossless multiple line and that each element of the latter matrix is a continuous function of x with a continuous first derivative. By hypothesis the matrix $C(x)$ has similar properties. Moreover, from the reality of these matrices and the form of (4.2) we note that the coefficient matrices in (4.2) are pure real. Hence, we have that the linear second order vector differential equation (4.2) is a special case of the general system of linear second order differential equations investigated by Sternberg [5]; we note in passing that since every complex solution $v = v(x)$ of (4.2) is a linear combination of real solutions of (4.2) the restriction of the treatment in [5] to consideration of only real solutions of the system of differential equations investigated will be seen to cause no essential difficulties for our purposes.

Consider now for fixed values of the angular frequency ω the question of the

existence or non-existence of *multiple voltage nodes* on the lossless multiple line, i.e., the question of the existence or non-existence of one or more points x_0 at which all of the voltages are zero for all time. In view of the form of the relations (2.2) between the instantaneous and the complex currents and voltages it is clear that the latter question is equivalent in the present case of a lossless multiple line to the question of the existence, or non-existence for fixed values of ω, of solutions $v = v(x) \not\equiv 0$ of equation (4.2) satisfying $v(x_0) = 0$ at one or more points x_0 on the lossless multiple line where zero stands for the null vector. For this form of the question we now note some known facts. In the first place, fundamental existence theorems for systems of differential equations tell us that for an arbitrarily selected point x_0 on the lossless multiple line there always exist n linearly independent solutions $v^1 = v^1(x), \cdots, v^n = v^n(x)$ of (4.2) which are equal to the null vector at x_0. On the other hand in view of Rolle's theorem and the same fundamental existence theorems mentioned above, it follows that the set of null points of a solution $v = v(x) \not\equiv 0$ of (4.2) cannot have a finite limit point so that we have the result that along a finite lossless multiple line there are only finitely many voltage nodes with a similar statement holding for currents. Finally, for a given lossless multiple line and equation (4.2) it may happen that no solution $v = v(x)$ of (4.2) can be equal to the null vector at more than one point x_0 without being identically null along the lossless multiple line, i.e., it may happen that the only solution $v = v(x)$ of (4.2) satisfying boundary conditions of the form $v(x_1) = 0 = v(x_2)$, $x_1 < x_2$, is the trivial solution $v(x) \equiv 0$. In such a case the vector differential equation (4.2) is said to be *non-oscillatory* on the interval of the x-axis considered, and the lossless multiple line will be said to be *non-nodal with respect to voltages at the angular frequency* ω. In every other case the equation (4.2) is said to be oscillatory on the interval considered; however, inasmuch as such an equation may be oscillatory on an interval and yet have one or more real or complex non-oscillatory solutions on that interval which represent the actual physical phenomena involved, the lossless multiple line will not be classified as nodal in this case. Finally, the equation (4.2) is said to be *non-oscillatory for large* x and the infinite lossless multiple line will be said to be *non-nodal with respect to voltages for large* x *at the angular frequency* ω if there exists a point $a_0 \geqq a$ beyond which no solution $v = v(x) \not\equiv 0$ of (4.2) is equal to the null vector twice; we note in passing that in such a case there can be only finitely many voltage nodes on the infinite lossless multiple line.

From the above definitions and the form of equation (4.2) it is evident that the non-nodal or converse character of a given lossless multiple line is in general dependent upon the angular frequency ω. However, it will be seen below that at least in one case of an infinite lossless multiple line the line will be non-nodal for large x at all angular frequencies ω. In connection with the proofs of the theorems about to be given it is to be noted that although as remarked earlier the treatment in Sternberg [5] of vector differential equations of the type (4.2) is restricted to consideration of only real solutions of such vector differential equations, we have from the fact that a complex solution $v = v(x)$ of (4.2) is a linear combination of real solutions of (4.2), that (4.2) is non-oscillatory on an interval or for large x in the sense defined above if and only if (4.2) is non-oscillatory on the

interval or for large x under the same definition when only real solutions of (4.2) are considered. We begin now with the following theorem; we refer always to a lossless multiple line having the voltage equation (4.2) with the matrices $C(x)$ and $L(x)$ satisfying the hypotheses stated earlier.

A finite lossless multiple line is non-nodal with respect to voltages at the angular frequency ω if and only if there exists a pure imaginary continuous symmetric matrix $H = H(x)$ satisfying the Riccati type matrix differential equation

$$H' - j\omega HL(x)H + j\omega C(x) = 0 \tag{4.3}$$

along the lossless multiple line. An infinite lossless multiple line is non-nodal with respect to voltages for large x at the angular frequency ω if and only if there exists such a matrix $H = H(x)$ satisfying (4.3) for large x.

The preceding theorem is an easy consequence of Theorem 4.2 of Sternberg [5; §4]; in particular, the first assertion follows readily from the theorem of [5; §4] mentioned, provided that the latter is suitably reformulated to apply on a finite interval while the second assertion follows immediately from the theorem as stated in [5; §4]. We note in passing that since the matrices $C(x)$ and $L(x)$ have been assumed to be positive definite along the lossless multiple line, it follows from the form of (4.3) that a pure imaginary continuous symmetric matrix $H = H(x)$ satisfying (4.3) as described above is necessarily a non-constant matrix; in fact, in the case of an infinite lossless multiple line if, for example, the matrix $I - L(x)$ is negative semi-definite along the lossless multiple line, then we have by Theorem 4.4 of [5; §4], that such a matrix $H = H(x)$ when it exists, satisfies the matrix limit relation $\lim_{x\to\infty} H(x) = 0$.

We next give a theorem which bears a certain similarity, particularly in its proof, to certain results noted by Eckart [2; pp.177–178] for a lossless simple line. We recall that the lossless multiple line is supposed to extend from $x = a$ to $x = b$ where $0 \leqq a < b \leqq \infty$.

If $a > 0$ and for a finite or infinite lossless multiple line each of the matrices

$$I - L(x), \qquad I - 4\omega^2 x^2 C(x), \tag{4.4}$$

is positive semi-definite along the line then the lossless multiple line is non-nodal with respect to voltages at the angular frequency ω and at all lower angular frequencies.

To establish the preceding theorem we begin by noting that the linear second order vector differential equation $Iv^{0\prime\prime} + [1/(4x^2)]Iv^0 = 0$ is non-oscillatory on every interval $a_0 \leqq x < \infty$ with $a_0 \geqq a > 0$. Hence employing Lemma 3.2 of [5; §3] we have that the integral

$$J_0[v;\, a_0\,,\, b_0] \equiv \int_{a_0}^{b_0} \left[v^{*\prime} v' - \frac{1}{4x^2}\, v^* v \right] dx \tag{4.5}$$

satisfies $J_0[v;\, a_0\,,\, b_0] > 0$ for arbitrary admissible variations $v = v(x) \not\equiv 0$ on $a_0 \leqq x \leqq b_0$, $a \leqq a_0 < b_0 < \infty$, satisfying $v(a_0) = 0 = v(b_0)$. Next introducing the integral

$$J[v;\, a_0\,,\, b_0] \equiv \int_{a_0}^{b_0} [v^{*\prime} L^{-1}(x) v' - \omega^2 v^* C(x) v]\, dx, \tag{4.6}$$

where $v = v(x)$ is as before, we note under the hypothesis that the matrices (4.4) are positive semi-definite along the lossless multiple line, that we have $J[v; a_0, b_0] \geqq J_0[v; a_0, b_0]$ for arbitrary admissible variations $v = v(x) \not\equiv 0$ satisfying $v(a_0) = 0 = v(b_0)$ where we take $b_0 \leqq b$ in case $b < \infty$, so that in any case we have $J[v; a_0, b_0] > 0$ for such $v = v(x)$. The conclusions of our theorem now follow readily from facts noted in the second paragraph of [5; §3] together with a second application of Lemma 3.2 of [5; §3].

We terminate our work with the following theorem of a somewhat different nature in which the conditions given are independent of the angular frequency ω. To establish this theorem one has only to apply Theorems 5.1 and 5.3 of [5; §5].

If for an infinite lossless multiple line the matrix $I - L(x)$ is negative semi-definite along the line then a necessary condition for the infinite lossless multiple line to be non-nodal for large x at all angular frequencies ω is that the improper matrix integral $\int_a^\infty x^r C(x)\,dx$ exists for each r satisfying $0 \leqq r < 1$. Conversely, if for an infinite lossless multiple line the matrix $I - L(x)$ is positive semi-definite along the line then a sufficient condition for the infinite lossless multiple line to be non-nodal for large x at all angular frequencies ω is that the improper matrix integral $\int_a^\infty xC(x)\,dx$ exists.

For further theorems along the above lines, and more general forms of some of the theorems given, we refer to [5; §§4 and 5].

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ON THE APPROXIMATION OF LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS BY DIFFERENCE EQUATIONS WITH POSITIVE COEFFICIENTS*

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1. Introduction. Consider a differential expression of the form

$$L[u] \equiv \tfrac{1}{2} \sum_{i,k=1}^{n} a_{ik}(x)\, \partial^2 u/\partial x_i\, \partial x_k + \sum_{j=1}^{n} b_j(x)\, \partial u/\partial x_j + c(x)u \tag{1}$$

where x is the point with coordinates $x_1, \cdots, x_n$ in real n-dimensional space, and with it a finite difference expression of the form

$$L_h[u] \equiv \sum_{s=0}^{N} c_s(x, h)\, u(x + m_s h), \tag{2}$$

where m_s is an n-dimensional vector with integral components m_{sj}, $(j = 1, \cdots, n)$, and h is the mesh length. Without loss of generality we can assume that $m_s \neq m_t$, for $s \neq t$, and that m_0 is the zero vector. The difference expression (2) shall be called a *formal approximation*, or *formally* analogous to $L[u]$, if the formal Taylor expansions of $u(x + m_s h)$ in (2) about the point x produce an expression whose beginning is proportional to $L[u]$, or, in other words, if there is a function $\lambda(x, h)$ not identically zero such that identically in x and h.

$$\sum_{s=0}^{N} c_s(x, h) = h^2 \lambda(x, h)\, c(x) \tag{3}$$

$$\sum_{s=1}^{N} c_s(x, h) m_{sj} = h\lambda(x, h) b_j(x), \qquad (j = 1, \cdots, n) \tag{4}$$

$$\sum_{s=1}^{N} c_s(x, h) m_{si} m_{sk} = \lambda(x, h) a_{ik}(x) \qquad (i, k = 1, \cdots, n). \tag{5}$$

In general, there will be infinitely many difference expressions that are formally analogous to a given differential expression (see [1]). In many applications it is desirable to deal with a formal approximation such that

$$c_s(x, h) > 0 \ (s \neq 0), \qquad c_0(x, h) < 0 \tag{6}$$

or, at least,

$$c_s(x, h) \geqq 0 \ (s \neq 0), \qquad c_0(x, h) \leqq 0. \tag{6a}$$

This is particularly true in the theory of boundary value problems for differential equations of *elliptic* type.

If (6) is satisfied in some bounded region, and if there $\sum_{s=0}^{N} c_s(x, h) \leqq 0$, then it is easy to prove that the solutions of the difference equation $L_h[u] = 0$ satisfy a maximum principle analogous to that for elliptic differential equations. This will be more fully discussed in section 3.

The existence of such a principle permits the application of standard methods to prove existence, uniqueness and convergence, as $h \to 0$, of the solutions of

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boundary value problems for the difference equation. It makes it possible to give useful estimates of the truncation error, [2], [3]. It can even be used in problems concerning the solution of non-linear differential equations by finite difference methods. Such an application occurs, e.g., in a recent paper by L. Bers [4]. As pointed out by Bers, theorem 2 of the present article makes possible an extension of his results to more general differential equations.

Finally, difference equations whose coefficients satisfy the conditions (6) have a natural relation to the theory of random walks, which can be exploited for the numerical integration of differential equations by sampling procedures [3].

The main purpose of this paper is to study the question whether a given differential expression of elliptic type is formally analogous to a difference expression $L_h[u]$ satisfying conditions (6) or (6a).

2. Difference expressions of positive type. In order to simplify the notation we shall frequently write c_s, λ, for $c_s(x, h)$, $\lambda(x, h)$, etc. A difference expression (2) will be said to be of *positive* or of *non-negative type*, if conditions (6) or (6a) are satisfied, respectively. The vectors $m_s (s = 1, \cdots, N)$ associate with every point x a set of "neighbors" $x + m_s h$. Thus we can speak of the *system of neighborhoods* defined in the lattice by these vectors.

Without condition (6) it is possible to find for every $L[u]$ an analogous $L_h[u]$ using some fixed universal neighborhood system. In fact, it is amply sufficient to take the set of all vectors m_s with components $+1$, -1 or 0 (cf. [3]). Our first theorem shows that this is no longer true, if $L_h[u]$ is to be of positive type.

THEOREM 1. *Let n be greater than one. Corresponding to any system of neighborhoods and any point x one can find differential expressions $L[u]$ elliptic at x which do not possess a formally analogous difference expression of non-negative type, using that neighborhood system.*

Proof. Consider the $n(n + 1)/2$-dimensional space Ω of all symmetric matrices $\{X_{ik}\}$ of order n, and denote by D the set in Ω of all positive definite matrices. D is a convex cone with vertex at the origin and bounded by the portion Γ of the surface $\text{Det}\ \{X_{ik}\} = 0$ which corresponds to the positive semi-definite matrices. D is not a polyhedral cone. This is clear for $n = 2$, where Γ is one nappe of a non-degenerate quadric cone. If $n > 2$, consider the intersection of the linear subspace defined by $X_{ik} = 0$, $(i, k > 2)$, with Γ. This intersection is the non-linear point set defined by the inequality $X_{11}X_{22} - X_{12}^2 \geqq 0$. Hence, Γ does not consist entirely of flat $(\frac{1}{2}n(n + 1) - 1)$-dimensional parts.

The N points $\{X_{ik}\} = \{m_{si}\, m_{sk}\}$, $(s = 1, \cdots, N)$, of Ω lie on Γ, since the matrix $\{m_{si}\, m_{sk}\}$ is positive semi-definite. All points in Ω with coordinates of the form $\sum_s^N c_s\, m_{si}\, m_{sk}$ where $c_s \geqq 0$, lie therefore in a convex polyhedral cone D_1 inscribed in D. Since D is not polyhedral, there are rays d in D that do not lie in D_1. It follows that, if the values of $a_{ik}(x)$ of the given differential expression $L[u]$ correspond at the given point x to a point on d, then the equations (5) are incompatible with the equation (6a), i.e. with the assumption that the difference expression was to be of non-negative type.

As a byproduct of our proof we see that a differential expression $L[u]$ with a matrix $\{a_{ik}(x)\}$ that is *indefinite* at a point x can never be formally analogous there to a difference expression of non-negative type.

If we drop the requirement that the same universal neighborhood system is to be used for all differential equations then the negative statement of theorem 1 can be replaced by the affirmative statement of theorem 2 below, even if the stronger conditions (6) instead of (6a) are imposed. We call $L[u]$ *uniformly elliptic* in the closure $\bar{B}$ of B, if it is elliptic and Det $\{a_{ik}(x)\} \geqq$ const. > 0 in $\bar{B}$.

THEOREM 2. *If the given differential expression $L[u]$ is uniformly elliptic in the closure $\bar{B}$ of a bounded region B of the x-space, and has uniformly bounded coefficients in $\bar{B}$, then there exists, for sufficiently small h, a difference expression $L_h[u]$ of positive type that is formally analogous to $L[u]$.*

We separate a preliminary lemma from the proof of the theorem proper.

The matrices $\{d_i\, d_k\}$ where d_j, $(j = 1, \cdots, n)$ are any n real numbers, not all zero, form the set of all non-negative definite nth order matrices of rank one. The corresponding points of Ω lie on Γ. We shall be especially concerned with matrices of the form $\{m_i\, m_k\}$, where the $m_j (j = 1, \cdots, n)$ are integers. These may be called non-negative matrices of rank one of *integral type*.

LEMMA. *Every closed bounded set $\bar{G}$ in D is contained in a convex polyhedral cone D_1 generated by points of Γ corresponding to non-negative definite matrices of rank one of integral type. The coefficients in the linear combination of these generating points which represent a point of $\bar{G}$ can be chosen so as to be uniformly bounded away from zero in $\bar{G}$.*

Proof. Since every positive quadratic form can be written as a sum of squares of n linear forms, every point of D lies in the interior of some polyhedral cone generated by points of Γ corresponding to non-negative matrices of rank one. By means of Borel's covering theorem we conclude that $\bar{G}$ lies in the interior of the join of a finite number of such polyhedral cones. The single polyhedral cone D^* which is generated by all the generating edges of these cones will therefore contain $\bar{G}$ in its interior. Every matrix $\{d_i\, d_k\}$ can be arbitrarily approximated by a matrix $\{m_i\, m_k\}$ of integral type, in the sense that the direction cosines of the rays in Ω corresponding to the two matrices can be made to differ as little as we please. We omit the proof of this fact, which is an easy consequence of the existence of a rational number m_j/n_j such that $| d_j - m_j/n_j | < 1/n_j^2$. Hence the cone D^* can be replaced by a cone with the properties required from D_1 in our lemma.

In order to prove the second assertion of the lemma, we show first that any interior point of a bounded convex polyhedron of m dimensions is a linear combination with positive coefficients of *all* vertices, or, as we shall say, admits a "*positive representation*". In fact, let P be that interior point and denote by Q some interior point with a positive representation. Let R be the intersection of the ray QP with the boundary of the polyhedron. A representation of P can be obtained by forming an appropriate linear combination with positive coefficients of the representation of Q with some non-negative representation of R. The resulting representation of P is therefore positive.

Now let P be any point of $\bar{G}$. Every hyperplane through P intersects D_1 in a convex polyhedron π. We show next that this hyperplane can be chosen in such a way that π is bounded, i.e., that the hyperplane is not parallel to any ray through the origin which lies in the closure $\bar{D}_1$ of D_1. To that end we observe

that the ray opposite to any ray of $\bar{D}_1$ does not lie in $\bar{D}_1$, since it corresponds to non-positive quadratic forms. This implies [5, pp. 59–60] the existence of a hyperplane through the origin that does not contain any ray of $\bar{D}_1$. The hyperplane through P parallel to this one will then have the desired property. Therefore P admits a positive representation in terms of the vertices of π, and by continuity, there is a whole neighborhood of P which admits a positive representation. Applying again Borel's covering theorem we conclude that the points of $\bar{G}$ admit a uniformly positive representation, as was to be proved.

Proof of theorem 2. If we identify the set $\bar{G}$ of the lemma with the set of points in Ω defined by the uniformly definite matrix $\{a_{ik}(x)\}$ of $L[u]$, as x ranges over $\bar{B}$, the lemma implies the existence of a neighborhood system m_s, $(s = 1, \cdots, N)$, as well as of N positive functions $k_s(x)$, uniformly bounded in $\bar{B}$, such that $\sum_{s=1}^{N} k_s\, m_{si}\, m_{sk} = a_{ik}$. If we choose c_s and a positive λ so that $k_s = c_s/\lambda$ and set $c_0 = -1$, then conditions (5) and (6) will be satisfied.

The value of λ to be taken we determine from equation (3), i.e.,

$$\lambda(x, h) = \left[\sum_{s=1}^{N} k_s(x) - h^2\, c(x)\right]^{-1}. \tag{7}$$

This expression will be uniformly positive, as required, if h is taken sufficiently small.

Finally, in order to satisfy the condition (4), we observe that a vector m_s generates the same matrix $\{m_{si}\, m_{sk}\}$ as its negative $-m_s$. We can assume, therefore, without loss of generality, that our neighborhood system contains with every vector m_s also the vector $-m_s$. The conditions (3), (5) and (6) will then not cease to be satisfied, if the c_s are changed in such a way that they remain positive and that the sum of each pair of coefficients c_s corresponding to opposite m_s is preserved. We change the notation in accordance with these remarks by writing μ_t $(t = \pm 1, \pm 2, \cdots, \pm \frac{1}{2}N)$ instead of m_s $(s = 1, \cdots, N)$, with $\mu_{-t} = -\mu_t$. The coefficients c_s may, in similar fashion, be renamed γ_t $(t = \pm 1, \cdots, \pm \frac{1}{2}N)$. Our calculations so far have imposed only conditions of the form

$$\gamma_t > 0, \qquad \gamma_t + \gamma_{-t} = C_t \qquad (t = 1, \cdots, \tfrac{1}{2}N) \tag{8}$$

where each C_t is the sum of a pair of c_s's corresponding to opposite m_s's, as determined previously.

The condition (4) can now be written

$$\sum_{t=1}^{N/2} (\gamma_t - \gamma_{-t})\mu_{tj} = \lambda h b_j \qquad (j = 1, \cdots, n) \tag{4a}$$

or, because of (8),

$$\sum_{t=1}^{N/2} (2\gamma_t - C_t)\mu_{tj} = \lambda h b_j .$$

In order to show that these equations for the γ_t have positive solutions we define $\delta_t = \delta_t(x, h)$ by the equations

$$2\gamma_t = C_t + \sqrt{C_t}\, \delta_t h, \qquad (t = 1, \cdots, \tfrac{1}{2}N) \tag{9}$$

which transforms (4a) into

$$\sum_{t=1}^{N/2} \sqrt{C_t}\mu_{tj}\, \delta_t = \lambda b_j. \tag{10}$$

Let $A = \{\sqrt{C_i}\mu_{ij}\}$ be the matrix of n rows and $N/2$ columns of the coefficients in this system. (Observe that, by our construction of D^*, we must have $N/2 \geqq n$.) The product AA^T is, in view of (5) and the definitions of C_i and μ_{ij}, precisely the matrix $\lambda\{a_{ik}\}$ whose determinant is, by assumption, uniformly bounded away from zero in $\bar{B}$. Hence, the matrix A has rank n and the system (10) possesses a solution which is uniformly bounded in $\bar{B}$. The corresponding values of γ_i in (9) are therefore positive, at least for sufficiently small h, and the same is true for the corresponding values of γ_{-i}, which, by (8) are given by

$$2\gamma_{-i} = C_i - \sqrt{C_i}\,\delta_i h.$$

This completes the proof of theorem 2.

3. The maximum principle. In order to illustrate the remarks of the introduction we conclude this paper with a discussion of the maximum principle mentioned there.

Hereafter all points mentioned will be assumed to belong to a given orthogonal lattice of mesh length h.

If the value of $c_s(x, h)$ for a certain point x is different from zero, we shall call the point $x + m_s h$ an *effective neighbor* of x in the neighborhood system defined by the $m_s (s = 1, \cdots, N)$. Consider a finite sequence of lattice points such that each point is an effective neighbor of its predecessor. Then each point of this sequence will be said to be *effectively linked* to all its subsequent points.

In theorem 4 it will be shown that in most cases of practical importance any point in a bounded domain B can be linked effectively to points outside B.

THEOREM 3. *Let x be restricted to the points of an orthogonal lattice of mesh length h in euclidean n-space E. Consider a solution $u(x)$ of the difference equation problem*

$$u(x) = \begin{cases} \sum_{s=1}^{N} c_s(x, h)\, u(x + m_s h), & \text{for } x \text{ in } B \\ v(x), & \text{for } x \text{ in } E - B \end{cases} \tag{11}$$

where

$$c_s(x, h) \geqq 0, \qquad (s = 1, \cdots, N), \qquad \sum_{s=1}^{N} c_s(x, h) \leqq 1, \quad \text{for } x \text{ in } B, \tag{12}$$

and $v(x)$ is a given bounded function in $E - B$, which is zero outside some bounded domain. If the maximum and the minimum of $u(x)$ in B are assumed at points that can be effectively linked to a point in $E - B$, then

$$\min v(x) \leqq u(x) \leqq \max v(x). \tag{13}$$

Proof. Since the method of our proof is the same for the two inequalities of (13), it suffices to prove the right hand inequality. For abbreviation we set $V = \max v(x)$, $U = \max_{x \in B} u(x)$. Then $V \geqq 0$, and $U \leqq V$ is certainly true, if $U \leqq 0$. Therefore we may assume that $U > 0$. Let x_0 be some point in B such that $u(x_0) = U$. Then

$$U = u(x_0) \leqq U \sum_{s_1(x_0)} c_s(x_0, h) + V \sum_{s_2(x_0)} c_s(x_0, h) \tag{14}$$

where the summations are to be extended over the values of s for which the points $x_0 + m_s h$ belong to those subsets $S_1(x_0)$ and $S_2(x_0)$ of the neighborhood of x_0 which are in B and in $E - B$, respectively. If

$$\sum_{S_2(x_0)} c_s(x_0, h) > 0 \tag{15}$$

then (12) and (14) show that the assumption $V < U$ leads to the contradiction

$$U < U \sum_{s=1}^{N} c_s(x, h) \leqq U \tag{16}$$

and (13) is proved. If $\sum_{S_2(x_0)} c_s(x_0, h) = 0$, we conclude from (11) and (12) that $u(x) = U$ at all effective neighbors x of x_0. The preceding argument can then be repeated at each point of a sequence that links x_0 to a point in $E - B$. At the last point in B of this sequence (15) holds and the theorem is proved by the contradiction in (16).

THEOREM 4. *If one of the following conditions is true, then every point of B can be effectively linked to a point in $E - B$.*

(a) *There exists a subset m_{s_j} $(j = 1, \cdots, M)$ of the vectors m_s such that the m_{s_j} are linearly independent and that $\sum_{j=1}^{M} c_{s_j}(x, h) > 0$ for all x in B.*

(b) *Conditions* (6) *are satisfied in B.*

(c) *The difference expression $\sum_{s=1}^{N} c_s(x, h)\, u(x + m_s h) - u(x)$ is of non-negative type; it is formally analogous to a differential expression $L[u]$ that is uniformly elliptic in $\bar{B}$, and h is sufficiently small.*

Proof. (a) Since at least one $c_{s_j}(x, h)$ is positive at a given point, every x possesses an effective neighbor in the set $x + m_{s_j} h$, $(j = 1, \cdots, M)$. Because of the assumed linear independence of the m_j a sequence of points each of which is an effective neighbor of the preceding one cannot contain the same point twice and must therefore link x effectively to a point in $E - B$.

(b) This is a special case of (a).

(c) Here we use an adaptation of the argument in [6], lemma 2. We show first that $\lambda(x, h)$ does not vanish in B. For $\lambda = 0$ implies $\sum_{s=1}^{N} c_s = 1$, because of (3) and since $c_0 \equiv -1$ here. But this would lead to a contradiction with (5) taken for $i = k$. If we now choose K so that $| m_{s1} | \leqq K$ for $s = 1, \cdots, N$, we obtain from (5)

$$\lambda a_{11} \leqq K\Big(\sum_{m_{s1}>0} c_s\, m_{s1} - \sum_{m_{s1}<0} c_s\, m_{s1}\Big).$$

We add the K-fold of relation (4), for $j = 1$, to this inequality and have

$$\lambda a_{11} + Kh\lambda b_1 \leqq 2K \sum_{m_{s1}>0} c_s\, m_{s1}.$$

If $b_1(x)$ is not identically zero we impose the restriction $h \leqq \frac{1}{2} K \min_{x,s} | a_{11}(x)/b_1(x) |$ and find (also for $b_1(x) \equiv 0$) that, in B

$$0 < \sum_{m_{s1}>0} c_s\, m_{s1}.$$

Hence, at least one term of this sum is positive, and every point x in B has an effective neighbor whose x_1-coordinate exceeds that of x by h or more. A finite sequence of such neighbors will link x effectively to a point in $E - B$. Observe that this proof does not make full use of the ellipticity of $L[u]$. Only the condition $a_{ii}(x) > 0$ in $\bar{B}$, for some i, was needed.

Remarks. 1. Assume that the maximum of $u(x)$ is attained at a point x_0 of B without effective neighbors in $E - B$. Then it follows from the proof of theorem 3 that $u(x)$ has its maximum value at every point of B to which x_0 can be effectively linked through points which have no effective neighbors in $E - B$. But it is not always true, not even if condition (c) of theorem 4 is satisfied, that this includes every point of B. A solution $u(x)$ may assume its maximum at a point in B, even a point without effective neighbors in $E - B$, and still not be a constant in B.

2. The conditions (12) and the assumption $c_0 = -1$ used in theorems 3 and 4 are, because of (7), equivalent to $c(x) \leqq 0$, i.e. the maximum principle has been proved here under essentially the same condition as that required for its validity in the case of a formally analogous elliptic differential equation.

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SIMULTANEOUS TRIGONOMETRIC APPROXIMATION

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1. Introduction. Network theory* gives rise to an approximation problem in which there is a question of simultaneous approximation, and in which the coefficients are sometimes subject to an inequality [1]. Specifically, one may wish to make both the expressions

$$I = \int_0^b \left[f(x) - \sum_0^n a_k^{(n)} \cos kx \right]^2 dx, \qquad J = \int_0^b \left[g(x) - \sum_0^n a_k^{(n)} \sin kx \right]^2 dx \quad (1)$$

small, where f and g are given real functions in L^2 and where the (real) a_k's are subject to

$$\sum_k [a_k^{(n)}]^2 \leqq M. \quad (2)$$

Here, the simultaneous approximation is dealt with by minimizing the sum $I + J$. Hence, with $\bar{F}(e^{-i\theta}) = F(e^{i\theta}) = f(\theta) + ig(\theta)$, the problem is to minimize

$$\int_{-b}^{b} \left| F(z) - \sum_{k=0}^{n} a_k^{(n)} z^k \right|^2 d\theta \quad (3)$$

on the arc $(-b, b)$ of the unit circle $z = e^{i\theta}$. This formulation, due to the Referee, reduces the problem to a simple case of polynomial approximation in the complex domain. The formulation first given, which is natural in applications, allows us to weight the two approximations differently by minimizing $I + kJ$ for constant k, and our techniques are applicable also to this situation.

The following notation is used throughout the paper:

$$c_k = \int_0^b f(x) \cos kx \, dx, \qquad s_k = \int_0^b g(x) \sin kx \, dx, \quad (4)$$

$$p_{jk} = \int_0^b (\cos jx \cos kx + \sin jx \sin kx) \, dx = \sin (j - k)b/(j - k) \text{ if } j \neq k$$

$$= b \qquad \text{if } j = k.$$

The symbols $\mathbf{c}$, $\mathbf{s}$, and $\mathbf{a}$ represent the column vectors whose components are c_k, s_k, and $a_k^{(n)}$ $(k = 0, 1, \cdots, n)$; the symbols $\mathbf{c}'$, $\mathbf{s}'$, and $\mathbf{a}'$ represent their transposes. By $\mathbf{P}$ we mean the $n + 1$ by $n + 1$ matrix (p_{jk}). In this notation

$$I + J = \mathbf{a}'\mathbf{P}\mathbf{a} - 2(\mathbf{c} + \mathbf{s})'\mathbf{a} + \int_0^b (f^2 + g^2) \, dx. \quad (5)$$

2. Approximation over $(0, \pi)$. Since I + J is a polynomial in the a_k's, a neces-

* It is a pleasure to acknowledge our indebtedness to Prof. E. A. Guillemin of M. I. T., for suggesting the problem, and for valuable assistance in its mathematical formulation.

sary condition for an extremum is that the derivative with respect to each a_k be zero. With (2) replaced by an equality, use of Lagrange multipliers gives

$$a_k = (t/\pi)(s_k + c_k), \tag{6}$$

$$t^2 = \frac{\pi^2 M}{\sum_{k=0}^{n} (c_k + s_k)^2}, \qquad t \geqq 0, \tag{7}$$

when $b = \pi$. If $t \geqq 1$, the constraint is ineffective. The fact that a minimum exists, the necessity of the conditions (6), (7), and the uniqueness of the solution with positive t, together show that the conditions are sufficient to ensure an absolute minimum.

It remains to show that (2) can be legitimately replaced by an equality. That will be the case if the prescribed M is less than the optimum M, and if the minimum error $I + J$ increases steadily as $\sum a_k^2$ deviates from its optimum value. Substituting in $I + J$, using the Parseval identity to express everything in terms of s_k, c_k, and collecting terms, gives us

THEOREM 1. *The minimum of* $\pi(I + J)$ *subject to* $a'a \leq M$ *is*

$$\pi(I + J)_{\min} = 2\sum_{n+1}^{\infty} (s_k^2 + c_k^2) + \sum_{0}^{n} (s_k - c_k)^2 + (t - 1)^2 \sum_{0}^{n} (s_k + c_k)^2. \tag{8}$$

The minimizing a *is given by* (6) *and* (7) *with* $t \geqq 0$.

Since $(t - 1)^2$ increases steadily, as t decreases, the use of an equality (2) was permissible. Similar considerations apply to the problems discussed below.

The first term of (8) is error due to use of only a finite number of terms in the approximating series. By Bessel's inequality, this term of the error tends to zero as n becomes infinite. The second term represents error due to the fact that the two functions f, g are not conjugate—the condition that they be conjugate is precisely that $c_k = s_k$ for all k. The presence of this term shows that the *error does not generally approach zero, no matter how large n becomes.* The third term represents error due to the constraint, and is to be taken as zero when $t \geqq 1$.

3. The case $0 < b < \pi$ without constraint. In contrast to the case $b = \pi$, it turns out that I and J can be made arbitrarily small simultaneously:

THEOREM 2. *If* f *and* g *are continuous in* (0, b) *with* $0 < b < \pi$, *there exist real constants* $a_k^{(n)}$ *such that*

$$f_n = \textstyle\sum_{k=0}^{n} a_k^{(n)} \cos kx, \qquad \text{and } g_n = \sum_{k=0}^{n} a_k^{(n)} \sin kx \tag{9}$$

converge uniformly to f *and* g *respectively in* $(0, b)$. *If* f *and* g *are in* $L^2(0, b)$, *the corresponding result for convergence in the mean is true.*

The result is an easy consequence of results in [7]. Another proof, due to R. Salem, is given in [1]. Considering the function $(f + ig)e^{-ims}$ shows, incidentally, that the approximation can be carried out using only the functions $\cos nx$, $\sin nx$ for $n \geqq m$.

The minimum of $I + J$ as given by (5) leads to the equation

$$\mathbf{Pa} - (\mathbf{c} + \mathbf{s}) = 0. \tag{10}$$

Now $\mathbf{P}$, as given by (4), is the sum of two Grammians, the first of which is positive definite and the second positive semidefinite. Hence $\mathbf{P}$ is positive definite (cf. also [2]). In particular, $\mathbf{P}$ has an inverse, and we can solve (10) to get

$$\mathbf{a} = \mathbf{P}^{-1}(\mathbf{c} + \mathbf{s}). \tag{11}$$

Substitution in (5) gives us

THEOREM 3. *If there is no constraint, then the minimizing* $\mathbf{a}$ *is given by* $\mathbf{a} = \mathbf{P}^{-1}(\mathbf{c} + \mathbf{s})$, *and the minimum value is*

$$(I + J)_{\min} = \int_0^b (f^2 + g^2)\, dx - (\mathbf{c} + \mathbf{s})'\mathbf{P}^{-1}(\mathbf{c} + \mathbf{s}). \tag{12}$$

The expression (12) can be written as the ratio of two determinants:

$$(I + J)_{\min} = \begin{vmatrix} \mathbf{P} & \mathbf{c} + \mathbf{s} \\ (\mathbf{c} + \mathbf{s})' & \int (f^2 + g^2)\, dx \end{vmatrix} |\mathbf{P}|^{-1}. \tag{13}$$

A direct proof that (13) tends to 0 as $n \to \infty$ when $0 < b < \pi$ would be of interest.

In the following theorem the subscripts f and g indicate the functions relative to which the integrations in (3) are carried out:

THEOREM 4. *If* $0 < b < \pi$, *then, as* $n \to \infty$,

$$\text{(a)}\quad (\mathbf{c}_f + \mathbf{s}_g)'\mathbf{P}^{-1}(\mathbf{c}_f + \mathbf{s}_g) \to \int_0^b (f^2 + g^2)\, dx$$

$$\text{(b)}\quad \mathbf{c}_f'\mathbf{P}^{-1}\mathbf{c}_f \to \int_0^b f^2\, dx$$

$$\text{(c)}\quad \mathbf{s}_g'\mathbf{P}^{-1}\mathbf{s}_g \to \int_0^b g^2\, dx$$

$$\text{(d)}\quad \mathbf{c}_f'\mathbf{P}^{-1}\mathbf{c}_g \to \int_0^b fg\, dx$$

$$\text{(e)}\quad \mathbf{s}_f'\mathbf{P}^{-1}\mathbf{s}_g \to \int_0^b fg\, dx$$

$$\text{(f)}\quad \mathbf{c}_f'\mathbf{P}^{-1}\mathbf{s}_g \to 0$$

Theorems 2 and 3 give (a), of which (b) and (c) are special cases. Conditions (d) and (e) are equivalent to (b) and (c); and (f) follows from (a), (b), and (c).

The relations (b), (c), (d), (e) are analogous to the ordinary Parseval identity, which one would expect them to become when $b = \pi$. Actually, however, they exhibit a discontinuity at $b = \pi$, since then the left sides must be multiplied by 2 for equality. The relation (f) has no counterpart in the case $b = \pi$.

A remarkable property satisfied by the sequences $\{c_k\}$, $\{s_k\}$ is that one can omit the first m terms of each sequence, and use the curtailed sequences without altering the validity of Theorem 4. This follows from Theorem 2, and from

the fact that the matrix $\mathbf{P}$ is unaltered by the omission of the functions $\cos kx$, $\sin kx$, $k = 0, 1, 2, \cdots, m$. As a special case, if we take $f = 1$, $g = 0$ and omit c_0 and s_0, we get, using (13),

$$\det (\mathbf{P}_{n+1})/\det (\mathbf{P}_n) \to 0. \tag{14}$$

4. Nonexistence of certain approximations. The next two theorems are concerned with the behavior of the approximating polynomials on (b, π).

THEOREM 5. *Suppose f and g have a series approximation in (a, b). If $f^* = f$ on a subinterval of (a, b), but $\int_a^b | f^* - f |^2 dx > 0$, then f^* and g do not admit a series approximation on (a, b). (Hence "in general" two functions f and g do not admit such an approximation.)*

The distinction between Theorem 5 and Theorem 2 is that in the case of a polynomial approximation $a_k^{(n)}$ depends on n whereas for a series approximation $a_k^{(n)}$ is independent of n (cf. [5]). To prove this theorem we need the following lemma:

LEMMA: *If $\lim_{n\to\infty} \int_a^b |\sum_0^n a_k e^{ikx}| dx = 0$ with $b - a > 0$, then each $a_k = 0$.* First, the a_k's are bounded. For, with $f_n = \sum_0^n a_k e^{ikx}$, we have $(b - a) | a_n | = \int_a^b | a_n e^{inx} | dx \leqq \int_a^b | f_n | dx + \int_a^b | f_{n-1} | dx$; so, in fact, $a_n \to 0$. Next, define $F_n(x) = \int_a^x f_n(t) e^{it} dt$, $\phi_n(x) = \int_a^x F_n(t) e^{it} dt$. Then, $F_n, \phi_n \to 0$ on (a, b). Integration gives

$$\phi_n(x) = -\sum_0^n \frac{a_k}{(k+1)(k+2)} e^{i(k+2)x} + \sum_0^n \frac{a_k}{(k+1)(k+2)} e^{i(k+2)a} + (e^{ix} - e^{ia}) \sum_0^n \frac{a_k}{k+1} e^{i(k+1)a}.$$

Since the a_k's are bounded, the first two series converge and hence so does the third. Thus, the expression obtained by replacing n by ∞ is zero throughout (a, b). But putting $w = e^{ix}$ shows that this is the boundary value of an analytic function which is evidently bounded in the unit circle $| w | \leqq 1$. Hence this function is identically 0, and therefore each a_k is 0.

A simple use of the lemma together with the Schwarz and Minkowski inequalities yields

THEOREM 6. *If it is possible to find a series approximation on (a, b) for which $I + J$ tends to 0, then the a_k's are uniquely determined.*

Applying Theorem 6 to a subinterval of (a, b), we get Theorem 5 at once. The connection with boundedness is exhibited by Theorem 7.

THEOREM 7. *The following classes of function-pairs f, g are equivalent:*

(a) *Those which admit a polynomial approximation with $I + J \to 0$ and with $\sum_{k=0}^n (a_k^{(n)})^2 \leqq M$;*

(b) *Those which admit a series approximation with $I + J \to 0$ and with*

$\sum_{k=0}^{\infty} a_k^2 \leqq M$. (*Hence "in general" the approximation described in* (a) *is not possible.*)

The following proof, much shorter than our original one, is due to the Referee: Suppose that there is a sequence approximation with the constraint. The constraint says that the approximating polynomials belong uniformly to class H^2. By weak compactness there is a subsequence converging (L^2 metric) to a power series of Hardy's class H^2, and by the integral formula for the coefficients the $a_k^{(n)}$ converge to the coefficients a_k. These latter furnish the desired series approximation.

Considering $\int_0^\pi (\sum a_k^{(n)} \sin kx)^2\, dx$ gives the following corollary:

COROLLARY: *Suppose f and g admit a polynomial approximation on $(0, b)$ with $I + J \to 0$ and with $|\sum a_k^{(n)} \sin kx| \leqq M$ on $(0, \pi)$, or $|\sum a_k^{(n)} \cos kx| \leqq M$ on $(0, \pi)$. Then f and g admit a series approximation with $I + J \to 0$ and with $\sum a_k^2 \leqq 2M^2$. (Hence an approximation bounded in this sense is generally not possible.)*

5. The case $b < \pi$ with constraint. The problem here is to minimize

$$I + J = \mathbf{a}'\mathbf{P}\mathbf{a} - 2\mathbf{A}'\mathbf{a} + \int_0^b (f^2 + g^2)\, dx \qquad (\text{with } \mathbf{A} = \mathbf{c} + \mathbf{s}) \quad (17)$$

subject to $\mathbf{a}'\mathbf{a} \leqq M$. By (11), it can be assumed that $M < \mathbf{a}'\mathbf{P}^{-2}\mathbf{A}$. Otherwise the constraint is ineffective.

THEOREM 8. *The equations*

$$(\mathbf{P} + z)\mathbf{a} = \mathbf{A} \quad (18)$$

and

$$\mathbf{A}'(\mathbf{P} + z)^{-2}\mathbf{A} = M \quad (19)$$

have a unique solution with z positive. This solution gives the absolute minimum of $I + J$ subject to $\mathbf{a}'\mathbf{a} \leqq M$; and the minimum value is

$$(I + J)_{\min} = \int_0^b (f^2 + g^2)\, dx - \mathbf{A}'(\mathbf{P} + 2z)(\mathbf{P} + z)^{-2}\mathbf{A}. \quad (20)$$

If we interpret $\mathbf{a}$ as being a point in $(n+1)$ dimensional Euclidean space, the equation "$I + J$ = constant" represents an ellipsoid with center at the point $\mathbf{P}^{-1}\mathbf{A}$. If the constant increases, the ellipsoid swells. The constraint $\mathbf{a}'\mathbf{a} \leqq M$ tells us that the point $\mathbf{a}$ must lie within a sphere of radius $M^{\frac{1}{2}}$ centered at the origin. The swelling ellipsoid, at a certain position, will first touch this sphere externally, and the point of contact $\mathbf{a}$ is the poiut we want. In particular we see that the minimum is attained on the sphere; that is, with $\mathbf{a}'\mathbf{a} = M$. The condition of external tangency gives $\mathbf{P}\mathbf{a} - \mathbf{A} = -z\mathbf{a}$ with $z > 0$. In other words, $(\mathbf{P} + z)\mathbf{a} = \mathbf{A}$, or $\mathbf{a} = (\mathbf{P} + z)^{-1}\mathbf{A}$; and substitution into $\mathbf{a}'\mathbf{a} = M$ and (17) gives (19) and (20). Since $\mathbf{A}'(\mathbf{P} + z)^{-2}\mathbf{A}$ is a monotonic decreasing function of z for $z > 0$, it follows that (19) has a unique solution for z with $z > 0$.

One can simplify the equations (18), (19), (20) by introducing the orthogonal matrix $\mathbf{O}$ which reduces $\mathbf{P}$ to diagonal form. If we set $\mathbf{OPO}^{-1} = \mathbf{P}^*$, $\mathbf{Oa} = \mathbf{a}^*$, $\mathbf{OA} = \mathbf{A}^*$, then $p^*_{jk} = \lambda_j$ (the j^{th} eigenvalue of P) for $j = k$, and $p^*_{jk} = 0$ for $j \neq k$; and (18), (19), (20) become:

$$a^*_j = A^*_j/(\lambda_j + z) \tag{21}$$

$$\sum_0^n A^{*2}_j/(\lambda_j + z)^2 = M \tag{22}$$

$$(I + J)_{\min} = \int_0^b (f^2 + g^2)\, dx - \sum_0^n (\lambda_j + 2z)(\lambda_j + z)^{-2} A^{*2}_j. \tag{23}$$

Then (22) and (23) give

$$\frac{d(I + J)_{\min}}{dM} = -z; \tag{24}$$

so that z, which was introduced merely as a factor of proportionality, plays a more fundamental role in the whole problem*. An estimate of z would give an estimate of $(I + J)_{\min}$. However, we have been unable to do this, and the main question remains open: How good an approximation can be obtained with given f, g, n, b, M?

It is worth mentioning that the equations (21), (22), (23) give a practicable way of carrying out the calculations. First one must diagonalize $\mathbf{P}$, and obtain the transformation matrix $\mathbf{O}$. Ordinarily one decides upon the range b and n once for all, so that the same $\mathbf{O}$ may be used repeatedly. Once $\mathbf{O}$ is found it is easy to get $\mathbf{A}^*$ and the λ_j. Plotting the left member of the equation (22) gives z, and this too for the whole range of values M. The equations (21) then give a^*_j, our knowledge of $\mathbf{O}$ now gives a_j, and (23) gives $(I + J)_{\min}$.

6. The eigenvalues of P. If one is to carry out the program outlined in the previous paragraph, one must know the eigenvalues of $\mathbf{P}$, and we close with a few facts concerning them.

THEOREM 9. *Let* $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ *be the eigenvalues of* $\mathbf{P}_n = (\sin(j - k)b/(j - k))$; $j, k = 0, 1, \cdots, n$; $0 < b < \pi$. *A superscript distinguishes between the eigenvalues of* $\mathbf{P}_n$ *and those of* $\mathbf{P}_{n+1}$. *Then*

(1) $\lambda_j(b)$ *is an increasing function of* b; *therefore* $0 < \lambda_j(b) < \pi$.

(2) $\lambda_j^{(n+1)} \leq \lambda_j^{(n)} \leq \lambda_{j+1}^{(n+1)}$; $j = 0, 1, 2, \cdots, n$.

(3) $\lambda_{n-j}(\pi - b) = \pi - \lambda_j(b)$.

(4) *the curves* $\lambda = \lambda_j(b)$ *for* $j = 0, 1, 2, \cdots, n - 1$ *are tangent to the* b *axis at* $(0, 0)$, *and the curve* $\lambda = \lambda_n(b)$ *has slope* $n + 1$ *there.*

(5) $\lim_{n\to\infty} \lambda_0^{(n)}(b) = 0$; $\lim_{n\to\infty} \lambda_n^{(n)}(b) = \pi$.

PROOF: The assertion (1) follows from the fact that $\mathbf{P}$ is the sum of two Grammians with b the upper limit of integration, and from Courant's minimax expression [3] for λ_j.

The first inequality of (2) also follows from the minimax principle [4]. One has

* A different derivation of (24) is given in [1]. Cf. also [6].

only to consider the quadratic form associated with $\mathbf{P}_n$ as being obtained from that of $\mathbf{P}_{n+1}$ by setting the last variable equal to zero. The second inequality of (2) is proved similarly using the maximin principle.

Let $\mathbf{E}$ denote the identity matrix. Then, multiplication of the even rows and odd columns of $\det(\mathbf{P}(b) - \lambda\mathbf{E})$ by -1 gives us $\det(\mathbf{P}(\pi - b) - (\pi - \lambda)\mathbf{E})$ since $\sin kb = (-1)^{k+1} \sin k(\pi - b)$; and this gives (3).

The result (4) follows if all functions involved are expanded in a power series about $b = 0$; and (5) follows from (14).

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COEFFICIENTS TO FACILITATE INTERPOLATION AND INTEGRATION OF LINEAR SUMS OF EXPONENTIAL FUNCTIONS

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1. Introduction. The function

$$f(x) = \sum_{j=0}^{n} C_j e^{-jx} \tag{1}$$

which assumes the values f_i when $x = x_i$, $i = 1, 2, \cdots n$ can be expressed in the form of a Lagrangian as

$$f(x) = \sum_{i=0}^{n} f_i K_i^{(n)}(x) \tag{2}$$

where

$$K_i^{(n)}(x) = \frac{\prod_{p=0}^{n}{}' (e^{-x} - e^{-x_p})}{\prod_{p=0}^{n}{}' (e^{-x_i} - e^{-x_p})} \tag{3}$$

Here the notation $\prod'$ signifies that the factor having $i = p$ is omitted. We can also write

$$K_i^{(n)}(x) = \sum_{j=0}^{n} A_{ij}^{(n)}(x) e^{-jx} \tag{4}$$

and so

$$f(x) = \sum_{j=0}^{n} e^{-jx} \{ \sum_{i=0}^{n} f_i A_{ij}^{(n)} \} \tag{5}$$

The elements of the matrix $(A_{ij}^{(n)})$ can be obtained by expanding (3). Another approach is to put in turn $f(x) = 1, e^{-x} \cdots e^{-nx}$, in (5). Then the elements $A_{ij}^{(n)}$ can be determined by solving n sets of linear equations, each set containing n equations and n unknowns. Assume that the points x_i are equally spaced so that $x_i - x_{i-1} = h$, $i = 1, 2, \cdots n$, and $x_0 = 0$. The computation is considerably shortened for then the matrix $(A_{ij}^{(n)})$ is symmetric. For the calculations, the following relationships are useful for check purposes.

$$\begin{aligned} \sum_{i=0}^{n} A_{ij}^{(n)} &= 1 \qquad \text{if } j = 0 \\ &= 0 \qquad \text{if } j = 1, 2, \cdots n \end{aligned} \tag{6}$$

For integration, we have

$$\int_{rh}^{sh} f(x)\, dx = \sum_{i=0}^{n} f_i B_i^{(n)}(s, r) \tag{7}$$

where

$$B_i^{(n)}(s, r) = \sum_{j=0}^{n} (1/j)(e^{-rjh} - e^{-sjh}) A_{ij}^{(n)} \tag{8}$$

A useful check for the integration coefficients is

$$\sum_{i=0}^{n} B_i^{(n)}(s, r) = h(s - r) \tag{9}$$

The interpolation coefficients $A_j^{(n)}$ are directly applicable to the set of decay functions $e^{-jx}, j = 0, 1, 2, \cdots n$. However, it is easy to see that they also apply to the set of growth functions $e^{jx}, j = 0, 1, 2 \cdots n$ or to the mixed set $j = -m, -m + 1, \cdots 0, 1, \cdots m - 1, m$ where $n = 2m$ and n is even. The latter system is called symmetrical. For example, if $n = 2$, express $e^{-2x}f(x)$ in terms of decay functions, multiply the result by e^{2x} whence $f(x)$ is expressed in terms of growth functions. Likewise, if the symmetrical arrangement is desirable, then one deals with $e^{-x}f(x)$. If $T_i^{(n)}$ are integration coefficients when $f(x)$ is expressed in terms of growth functions, then $T_i^{(n)}(s, r) = B_{n-i}^{(n)}(n - r, n - s)$. A simple correlation does not seem possible when $f(x)$ is in the symmetrical form.

It is to be recognized that the problem of interpolation with exponentials is akin to that of trigonometric interpolation since, unlike that of Lagrangian polynomial interpolation, the coefficients depend on h. If an interpolation formula is linear in the f_i's, if the coefficient of f_i is separable into products as in (3), and if the sum of the coefficients of f_i is a constant, Salzer[1] has given a scheme for computing interpolation coefficients. Let $D_i^{(n)}$ be the reciprocal of the denominator of (3). Defining

$$a_i^{(n)} = D_i^{(n)}/(e^{-x} - e^{-x_i}) \tag{10}$$

it can readily be shown that

$$f(x) = \sum_{i=0}^{n} a_i^{(n)} f_i / \sum_{i=0}^{n} a_i^{(n)} \tag{11}$$

In the reference cited, Salzer has applied this technique to evaluate coefficients to facilitate trigonometric interpolation. For the present problem, though this scheme would reduce the number of basic coefficients which need to be computed, it would leave much to be desired from a practical point of view, as much computation would remain. Furthermore, (11) is not in a form suitable for integration purposes.

2. Analysis of the Remainder. The coefficients $A_j^{(n)}$ and $B_i^{(n)}$ are such that (5) and (7) are exact whenever $f(x)$ has the form prescribed by (1). When $f(x)$ is not of this form, it is of interest to examine the remainder when employing the interpolation and integration coefficients. If we consider the $(n + 1)$ point Lagrangian

$$g(u) = \sum_{i=0}^{n} g(u_i) \left\{ \frac{\prod_{p=0}^{n}{}' (u - u_p)}{\prod_{p=0}^{n}{}' (u_i - u_p)} \right\} + R_n(u) \tag{12}$$

then

$$R_n(u) = \left\{ \prod_{i=0}^{n} (u - u_i) \right\} \frac{[g^{(n+1)}(u)]_{u=\theta}}{(n + 1)!} \tag{13}$$

where θ lies in the interval containing u and the points u_i. Impose the transfor-

mation $u = e^{-x}$ and let $g(u) = f(x)$. Then $f(x)$ is of the form (2) and the remainder for the interpolation formula becomes

$$R_n(x) = \left[\prod_{i=0}^{n} (e^{-x} - e^{-x_i})\right] \frac{[(-e^x\, d/dx)^{n+1} f(x)]_{x=\xi}}{(n+1)!} \tag{14}$$

where ξ lies in the interval containing x and the points x_i. The remainder formulae in their present form are not too convenient, since for applications it is often difficult or impossible to evaluate the necessary derivatives. For numerical purposes, we can often get an estimate of the error if $g^{(n+1)}(u)/(n+1)!$ is replaced by the $(n+1)$th divided difference of $g(u)$. This technique is only approximate and should be used with caution. However, it usually comes near the true situation if the divided differences are formed for sufficiently small interval.[3]

Let $I_n(s, r)$ be the remainder when integration coefficients are employed. Then following Milne[3],

$$I_n(s, r) = H_n(s, r) \frac{[(-e^x d/dx^{n+1} f(x)]_{x=\xi}}{(n+1)!} \tag{15}$$

where

$$H_n(s, r) = \frac{e^{-(n+1)r\lambda} - e^{-(n+1)s\lambda}}{(n+1)} - \sum_{i=0}^{n} B_i^{(n)}(s, r) e^{-i\lambda(n+1)} \tag{16}$$

Equations (14)–(16) apply to the case of decay functions. The relations are easily modified if growth functions are employed. Simply replace x by its negative in Equation (14). The same is done in (15) save that the equation corresponding to (16) becomes

$$\bar{H}_n(s, r) = \frac{e^{(n+1)s\lambda} - e^{(n+1)r\lambda}}{(n+1)} - \sum_{i=0}^{n} B_{n-i}^{(n)}(n-r, n-s) e^{i\lambda(n+1)} \tag{17}$$

If $r + s = n$, it is readily shown that

$$\bar{H}_n(s, r) = e^{n\lambda(n+1)} H_n(s, r) \tag{18}$$

Greenwood[4] has given error formulae for the growth and symmetric integration coefficients. Application of his formulae to a simple numerical example yields results much too large to be of practical value. Employing our error formulae gives more realistic bounds. Brock and Murray[5] in a recently published paper discuss the error when integrating $e^{\lambda t}$ for a t step of length h using exponential sums. The case for an arbitrary integrand is also studied.

3. Description of the Tables. The present tables give values of $A_{ij}^{(n)}$ for $n = 2$, 3 and 4 (i.e., 3, 4 and 5 points) and $h = 0.1(0.1)1.0$. If $n = 2$ or 3, tabulations are also given for $h = 0.02$ and 0.05. Ten significant figures were carried in all the calculations. The coefficients were then rounded to eight significant figures and are so presented in Table I. Relation (6) was used as a check. Viewed as a function of h, the coefficients $A_{ij}^{(n)}$ do not difference well as they are singular at the origin. The singularity is removed if $h^n A_{ij}^{(n)}$ is formed. If $n = 2$, and six

TABLE I*

Interpolating coefficients $A_{ij}^{(n)} = A_{ji}^{(n)}$

$n = 2$ (Three Point)

	$h = 0.02$	$h = 0.05$	$h = 0.1$	$h = 0.2$
A_{00}	0.12129 558(4)	0.18545 209(3)	0.42945 860(2)	0.91834 664(1)
$-A_{01}$	.24999 167(4)	.39991 668(3)	.99916 708(2)	.24916 833(2)
A_{02}	.12879 608(4)	.21546 459(3)	.57970 848(2)	.16733 367(2)
A_{11}	.51018 569(4)	.84189 296(3)	.22195 525(3)	.62088 380(2)
$-A_{12}$	.26019 402(4)	.44197 628(3)	.12203 854(3)	.37171 547(2)
A_{22}	.13139 794(4)	.22651 169(3)	.64067 695(2)	.20438 180(2)
	$h = 0.3$	$h = 0.4$	$h = 0.5$	$h = 0.6$
A_{00}	0.34767 432(1)	0.16590 591(1)	0.89711 365	0.52427 065
$-A_{01}$	.11028 151(2)	.61673 291(1)	.39176 981(1)	.26959 233(1)
A_{02}	.85514 083(1)	.55082 701(1)	.40205 844(1)	.31716 526(1)
A_{11}	.31122 754(2)	.19892 972(2)	.14567 106(2)	.11646 704(2)
$-A_{12}$	.20094 602(2)	.13725 643(2)	.10649 408(2)	.89507 805(1)
A_{22}	.11543 194(2)	.82173 733(1)	.66288 231(1)	.57791 279(1)
	$h = 0.7$	$h = 0.8$	$h = 0.9$	$h = 1.0$
A_{00}	0.32287 048	0.20641 526	0.13567 635	0.91089 622(−1)
$-A_{01}$	.19594 856(1)	.14817 671(1)	.11545 041(1)	.92067 359
A_{02}	.26366 152(1)	.22753 518(1)	.20188 277(1)	.18295 840(1)
A_{11}	.99055 917(1)	.88210 076(1)	.81388 468(1)	.77235 824(1)
$-A_{12}$	.79461 061(1)	.73392 404(1)	.69843 427(1)	.68029 088(1)
A_{22}	.53094 909(1)	.50638 886(1)	.49655 150(1)	.49733 249(1)

$n = 3$ (Four Point)

	$h = 0.02$	$h = 0.05$	$h = 0.1$
$-A_{00}$	0.19615 517(5)	0.11459 385(4)	0.12275 198(3)
A_{01}	.61256 291(5)	.38025 405(4)	.45128 936(3)
$-A_{02}$	.63756 208(5)	.42024 572(4)	.55120 606(3)
A_{03}	.22116 434(5)	.15468 552(4)	.22366 868(3)
$-A_{11}$	.19005 667(6)	.12422 919(5)	.16116 723(4)
A_{12}	.19649 905(6)	.13502 938(5)	.19044 333(4)
$-A_{13}$	.67698 672(5)	.48825 587(4)	.74405 036(3)
$-A_{22}$	.20180 912(6)	.14433 373(5)	.21755 301(4)
A_{23}	.69066 275(5)	.51328 928(4)	.82230 282(3)
$-A_{33}$	.23484 038(5)	.17971 893(4)	.30192 114(3)
	$h = 0.2$	$h = 0.3$	$h = 0.4$
$-A_{00}$	0.11170 486(2)	0.23819 784(1)	0.71507 563
A_{01}	.50662 021(2)	.13414 304(2)	.50323 322(1)
$-A_{02}$	.75578 854(2)	.24442 455(2)	.11199 661(2)
A_{03}	.37087 319(2)	.14410 130(2)	.78824 048(1)
$-A_{11}$	.21855 310(3)	.70850 623(2)	.32939 925(2)
A_{12}	.30560 472(3)	.11755 506(3)	.65091 778(2)
$-A_{13}$	.13771 365(3)	.60118 739(2)	.37184 185(2)
$-A_{22}$	.39822 970(3)	.17426 441(3)	.10936 440(3)
A_{23}	.16820 383(3)	.81151 810(2)	.55472 286(2)
$-A_{33}$	.67577 501(2)	.35443 200(2)	.26170 505(2)

* In all tables the number N in parentheses after an entry denotes the factor 10^N.

TABLE I—*Continued*

	$h = 0.5$	$h = 0.6$	$h = 0.7$
$-A_{00}$	0.25766 622	0.10382 322	0.45054 817(−1)
A_{01}	.22800 090(1)	.11619 773(1)	.64136 084
$-A_{02}$	.61977 071(1)	.38579 006(1)	.26008 465(1)
A_{03}	.51753 643(1)	.37997 465(1)	.30045 404(1)
$-A_{11}$	.18696 008(2)	.12049 431(2)	.84796 690(1)
A_{12}	.44192 195(2)	.34226 392(2)	.29077 262(2)
$-A_{13}$	.27776 196(2)	.23338 939(2)	.21238 954(2)
$-A_{22}$	.83789 693(2)	.72894 811(2)	.69246 417(2)
A_{23}	.45795 206(2)	.42526 319(2)	.42770 002(2)
$-A_{33}$	.23194 374(2)	.22987 127(2)	.24535 588(2)

	$h = 0.8$	$h = 0.9$	$h = 1.0$
$-A_{00}$	0.20593 797(−1)	0.97751 423(−2)	0.47727 042(−2)
A_{01}	.37484 314	.22863 063	.14410 166
$-A_{02}$	.18566 102(1)	.13831 347(1)	.10647 752(1)
A_{03}	.25023 609(1)	.21642 792(1)	.19254 463(1)
$-A_{11}$	.63634 154(1)	.50137 278(1)	.41032 361(1)
A_{12}	.26454 314(2)	.25365 771(2)	.25345 717(2)
$-A_{13}$	.20465 742(2)	.20580 673(2)	.21386 583(2)
$-A_{22}$	.70145 051(2)	.74602 924(2)	.82415 701(2)
A_{23}	.45547 347(2)	.50620 288(2)	.58134 759(2)
$-A_{33}$	.27583 965(2)	.32203 894(2)	.38673 623(2)

$n = 4$ (Five Point)

	$h = 0.1$	$h = 0.2$	$h = 0.3$
A_{00}	0.24958 482(3)	0.91147 391(1)	0.10266 631(1)
$-A_{01}$	.12899 185(4)	.61623 722(2)	.91903 777(1)
A_{02}	.24896 079(4)	.15367 031(3)	.29731 050(2)
$-A_{03}$	.21267 161(4)	.16751 064(3)	.41188 415(2)
A_{04}	.67844 187(3)	.67349 318(2)	.20621 080(2)
A_{11}	.65309 688(4)	.40298 737(3)	.79054 148(2)
$-A_{12}$	.12315 728(5)	.96336 878(3)	.24170 092(3)
A_{13}	.10247 365(5)	.99480 692(3)	.30858 750(3)
$-A_{14}$	.31726 876(4)	.37280 180(3)	.13675 035(3)
A_{22}	.22658 303(5)	.21925 811(4)	.68671 456(3)
$-A_{23}$	.18372 907(5)	.21440 166(4)	.80247 530(3)
A_{24}	.55407 244(4)	.76113 403(3)	.32773 061(3)
A_{33}	.14534 938(5)	.19960 095(4)	.87142 782(3)
$-A_{34}$	.42826 803(4)	.67928 916(3)	.33635 160(3)
A_{44}	.12362 017(4)	.22360 761(3)	.12475 026(3)

	$h = 0.4$	$h = 0.5$	$h = 0.6$
A_{00}	0.18089 293	0.40329 309(−1)	0.10358 315(−1)
$-A_{01}$	.21689 994(1)	.65485 716	.23011 058
A_{02}	.91385 454(1)	.36069 212(1)	.16628 044(1)
$-A_{03}$	.16026 858(2)	.79777 934(1)	.46218 947(1)
A_{04}	.98764 194(1)	.59854 001(1)	.41788 425(1)
A_{11}	.24940 652(2)	.10208 586(2)	.49227 421(1)
$-A_{12}$	.98376 204(2)	.52370 571(2)	.33081 398(2)
A_{13}	.15498 610(3)	.10176 520(3)	.79336 727(2)
$-A_{14}$	.79381 549(2)	.58948 363(2)	.50947 960(2)
A_{22}	.35230 654(3)	.23880 151(3)	.19403 262(3)
$-A_{23}$	.48726 053(3)	.38696 909(3)	.36466 209(3)
A_{24}	.22419 164(3)	.19693 122(3)	.20204 806(3)
A_{33}	.61185 731(3)	.55736 991(3)	.59816 445(3)
$-A_{34}$	.26355 602(3)	.26418 824(3)	.30821 720(3)
A_{44}	.10886 951(3)	.12021 998(3)	.15293 826(3)

TABLE I—*Continued*

	$h = 0.7$	$h = 0.8$	$h = 0.9$	$h = 1.0$
A_{00}	0.29171 802(−2)	0.87512 040(−3)	0.27459 627(−3)	0.89046 062(−4)
$-A_{01}$	.89498 414(−1)	.37397 640(−1)	.16472 266(−1)	.75503 069(−2)
A_{02}	.85128 519	.46966 734	.27390 718	.16665 611
$-A_{03}$	.29637 804(1)	.20418 420(1)	.14827 861(1)	.11205 649(1)
A_{04}	.31990 765(1)	.26086 971(1)	.22250 766(1)	.19613 700(1)
A_{11}	.26550 614(1)	.15523 289(1)	.96408 241	.62722 470
$-A_{12}$	.23516 387(2)	.18214 285(2)	.15047 795(2)	.13066 167(2)
A_{13}	.69689 146(2)	.66790 904(2)	.68367 538(2)	.73627 263(2)
$-A_{14}$	.48738 322(2)	.50091 550(2)	.54267 354(2)	.61180 770(2)
A_{22}	.17917 378(3)	.18192 017(3)	.19861 690(3)	.22949 321(3)
$-A_{23}$	.38671 867(3)	.44684 251(3)	.55072 272(3)	.71338 857(3)
A_{24}	.23020 999(3)	.28266 696(3)	.36687 971(3)	.49679 487(3)
A_{33}	.71799 873(3)	.93426 143(3)	.12913 216(4)	.18697 305(4)
$-A_{34}$	.39800 542(3)	.55216 798(3)	.80748 367(3)	.12288 486(4)
A_{44}	.21333 467(3)	.31698 388(3)	.49264 624(3)	.79127 315(3)

point Lagrangian polynomial interpolants are used, the latter functions can be interpolated to at least six decimal accuracy. In this connection we remark that $\lim_{h\to 0} h^2 A_{02}^{(2)} = \frac{1}{2}$ and $\lim_{h\to 0} h^2 A_{01}^{(2)} = \lim_{h\to 0} h^2 A_{12}^{(2)} = -1.0$. In view of (6), these are the only quantities that need be subtabulated. If $n = 3$ or 4, the differences of the functions $h^n A_{ij}^{(n)}$ decrease so slowly that it is hardly possible to interpolate with any sufficient accuracy unless very high order interpolants are used.

In Table II, the integration coefficients $B_i^{(n)}(n, 0) = B_i^{(n)}$ are presented for the same range of n and h as above. These were calculated using (8) and the $A_{ij}^{(n)}$'s to ten significant figures. It is apparent that since the interpolation coefficients are quite large in many cases, some loss of significant figures occurs in the evaluation of the integration coefficients. For $n = 2$, the $B_i^{(n)}$'s are given to eight decimals. Here a perfect check was obtained using (9). For $n = 3$ and 4, as many significant figures as could be obtained are presented. Employing (9) and $n = 3, 4$, the sum of the coefficients differed from the true value by not more than three units of the last decimal place given. On this basis it seems highly probable that these integration coefficients are in error by not more than two units of the last decimal place tabulated.

If $n = 2$, the interpolation coefficients can be interpolated to virtually full accuracy using six point Lagrangians. For $n = 3$, nine point Lagrangians are necessary to accomplish the above, while if $n = 4$, the differences decrease quite slowly and interpolation to a high degree of accuracy is not possible. The integration coefficients for $h = 0.02$ and 0.05 were obtained using high order interpolants.

The error induced using the interpolating formula can be bounded by evaluating the maximum (minimum) points of $\prod_{i=0}^{n}(e^{-x} - e^{-x_i})$. Let $G_n[kh, (k+1)h]$ be the largest value in magnitude of the latter product in the

TABLE II

Integrating coefficients $B_i^{(n)}(n, 0) = B_i^{(n)}$

$n = 2$ (Three Point)

h	$B_0^{(2)}$	$B_1^{(2)}$	$B_2^{(2)}$
0.02	0.00666 605	0.02666 790	0.00666 605
.05	.01665 716	.06668 610	.01665 674
.10	.03325 885	.13348 897	.03325 218
.20	.06609 635	.26791 391	.06598 974
.30	.09815 909	.40422 126	.09761 965
.40	.12916 249	.54337 859	.12745 892
.50	.15888 490	.68638 538	.15472 972
.60	.18716 139	.83428 407	.17855 454
.70	.21387 745	.98817 155	.19795 100
.80	.23896 268	1.14921 107	.21182 625
.90	.26238 451	1.31864 488	.21897 061
1.00	.28414 226	1.49780 742	.21805 032

$n = 3$ (Four Point)

h	$B_0^{(3)}$	$B_1^{(3)}$	$B_2^{(3)}$	$B_3^{(3)}$
0.02	0.00741 15	0.02276 87	0.02222 81	0.00759 15
.05	.01821 16	.05791 55	.05453 29	.01933 99
.10	.03543 20	.11909 87	.10548 40	.03998 54
.20	.06729 46	.25119 53	.19534 80	.08616 22
.30	.09616 524	.39682 365	.26583 252	.14117 860
.40	.12238 268	.55752 270	.31089 924	.20919 540
.50	.14615 726	.73576 972	.32148 656	.29658 645
.60	.16763 932	.93497 682	.28447 528	.41290 859
.70	.18695 872	1.15954 379	.18126 089	.57223 657
.80	.20424 517	1.41496 427	−.01420 494	.79499 548
.90	.21963 611	1.70798 495	−.33809 210	1.11047 106
1.00	.23327 742	2.04682 153	−.84034 245	1.56024 352

$n = 4$ (Five Point)

h	$B_0^{(4)}$	$B_1^{(4)}$	$B_2^{(4)}$	$B_3^{(4)}$	$B_4^{(4)}$
0.1	0.03067 0	0.14420 8	0.04998 7	0.14472 5	0.03041 0
.2	.05932 59	.29913 99	.07878 19	.30785 14	.05490 08
.3	.08517 35	.47357 13	.05934 17	.52161 75	.06029 61
.4	.10804 36	.67619 08	−.04855 67	.84704 75	.01727 47
.5	.12803 10	.91710 90	−.30895 91	1.40098 56	−.13716 65
.6	.14534 30	1.20902 66	−.82638 39	2.40661 89	−.53460 47
.7	.16023 42	1.56837 61	−1.77267 12	4.28753 64	−1.44347 52
.8	.17297 52	2.01664 33	−3.43081 76	7.84220 79	−3.40100 91
.9	.18383 46	2.58204 35	−6.26687 84	14.56663 29	−7.46563 27
1.0	.19306 74	3.30172 28	−11.04814 17	27.24939 34	−15.69604 26

interval from $x = kh$ to $x = (k + 1)h$, $k = 0, 1, 2, \cdots n - 1$. If $\bar{G}_n[kh, (k + 1)h]$ is the corresponding quantity for growth functions, then

$$\bar{G}_n[kh, (k + 1)h] = e^{nh(n+1)}G_n[(n - 1 - k)h, (n - k)h].$$

The quantities $G_n[kh, (k+1)h]$ are presented mostly to two significant figures in Table III for $n = 2, 3, 4$ and $h = 0.1(0.1)1.0$. To facilitate appraisal of the

TABLE III

Coefficients $G_n[kh, (k+1)h]$

$n = 2$

h	$-G_2(0, h)$	$G_2(h, 2h)$
0.1	0.31(−3)	0.26(−3)
.2	.20(−2)	.14(−2)
.3	.56(−2)	.33(−2)
.4	.11(−1)	.54(−2)
.5	.18(−1)	.74(−2)
.6	.25(−1)	.89(−2)
.7	.34(−1)	.99(−2)
.8	.42(−1)	.10(−1)
.9	.50(−1)	.10(−1)
1.0	.59(−1)	.10(−1)

$n = 3$

h	$-G_3(0, h)$	$G_3(h, 2h)$	$-G_3(2h, 3h)$
0.1	0.69(−4)	0.31(−4)	0.44(−4)
.2	.77(−3)	.28(−3)	.32(−3)
.3	.28(−2)	.78(−3)	.72(−3)
.4	.63(−2)	.14(−2)	.10(−2)
.5	.11(−1)	.19(−2)	.12(−2)
.6	.17(−1)	.23(−2)	.11(−2)
.7	.23(−1)	.25(−2)	.10(−2)
.8	.30(−1)	.25(−2)	.81(−3)
.9	.37(−1)	.23(−2)	.62(−3)
1.0	.43(−1)	.21(−2)	.46(−3)

$n = 4$

h	$-G_4(0, h)$	$G_4(h, 2h)$	$-G_4(2h, 3h)$	$G_4(3h, 4h)$
0.1	0.20(−4)	0.06(−4)	0.05(−4)	0.09(−4)
.2	.38(−3)	.84(−4)	.48(−4)	.73(−4)
.3	.17(−2)	.28(−3)	.12(−3)	.14(−3)
.4	.43(−2)	.52(−3)	.17(−3)	.16(−3)
.5	.82(−2)	.72(−3)	.18(−3)	.13(−3)
.6	.13(−1)	.83(−3)	.16(−3)	.91(−4)
.7	.18(−1)	.84(−3)	.12(−3)	.55(−4)
.8	.24(−1)	.78(−3)	.86(−4)	.31(−4)
.9	.29(−1)	.67(−3)	.56(−4)	.16(−4)
1.0	.34(−1)	.56(−3)	.35(−4)	.08(−4)

error when employing integration coefficients, the values of $H_n(n, 0) = H_n$ are given to two significant figures in Table IV for the same range as above.

The tables given here are new, save that Greenwood[4] has given integration coefficients as applied to growth functions for $n = 1(1)6$ for the special case

$h = 2/n$. He also gives integration coefficients for the symmetrical case when $n = 2, 4, 6$.

TABLE IV

Coefficients $H_n(n, 0) = H_n$

h	$-H_2$	$-H_3$	$-H_4$
0.1	0.30(−5)	0.51(−5)	0.58(−6)
.2	.71(−4)	.10(−3)	.31(−4)
.3	.40(−3)	.49(−3)	.24(−3)
.4	.13(−2)	.14(−2)	.84(−3)
.5	.30(−2)	.30(−2)	.20(−2)
.6	.57(−2)	.53(−2)	.38(−2)
.7	.95(−2)	.83(−2)	.61(−2)
.8	.14(−1)	.12(−1)	.88(−2)
.9	.20(−1)	.16(−1)	.12(−1)
1.0	.27(−1)	.20(−1)	.15(−1)

4. Applications. The tables have a wide variety of applications. They were particularly developed to aid in the numerical evaluation of integrals of the type $\int^{z} e^{iax} f(x)\,dx$. Here $f(x)$ is a rather complicated function whose analytic representation may or may not be known. Tabulation of the integrand followed by numerical integration is quite laborious when the integral is required for several values of z, some of which are quite large. For computational purposes, representation of $f(x)$ by a polynomial may sometimes be less convenient than representation by sums of exponentials. The latter is easily accomplished using the tables in this paper. The integration coefficients have been tested by integrating a number of known integrals and the results obtained are quite satisfactory.

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RAPIDLY CONVERGING SOLUTIONS TO INTEGRAL EQUATIONS

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1. Suppose that for a given known kernel $K(x, z)$ and function $F(x, z)$, we wish to find the unknown solution, $U(x, z)$, defined by the Fredholm integral equation

$$E(x, z) = F(x, z) + U(x, z) + \int_a^b K(x, s)U(s, z)\,ds = 0. \tag{1}$$

Important special cases met in practical applications often involve one or more of the following: $F(x, z)$ may be a function of the single variable x, often written $F = -f(x)$; the kernel K may involve a multiplicative parameter, λ, which is not explicitly shown here; K may vanish for $z > x$, and hence be of Volterra type with upper limit on the integral written as x rather than b; K may be of Poisson or Wiener type, of the form $K(x - z)$, with K vanishing for negative arguments. Throughout we assume that (1) does have a unique solution, which it will have for almost all K's, the only exception being when K happens to have an eigenvalue, $\lambda = 1$.

E as defined above is a measure of the error with which any arbitrary function inserted in place of the true solution U does satisfy the integral equation. E is a linear functional of each of the functions involved and for simplicity we can denote it by $E(F, U, K)$. It will be noted that the order of the functions is important, and that equating E to zero defines F explicitly in terms of the remaining two functions, whereas U is defined only implicitly in terms of the remaining two functions. To simplify notation, we write any integral of the form $\int_a^b K(x, s)U(s, z)\,ds$ as $K(x, z)\cdot U(x, z)$ or simply as $K\cdot U$. Similarly the definition of $K\cdot U\cdot W$ will be apparent.

If any approximation u to U were at hand, it would be natural to compute the error in satisfying (1), i.e. $E(F, u, K)$; and for many purposes, particularly where (1) is "ill-conditioned", it may be of more physical importance to know that E is small than that u-U is small. Moreover, many of the usual methods for approximate iteration to a solution U try to utilize knowledge of E to get an improved approximation. Ways of speeding up such approximating sequences, so that they will converge more rapidly, will be developed below.

2. It can be easily verified by substitution that the solution U can be written down in terms of F by means of the reciprocal kernel to K, $k(x, z)$; i.e. as

$$E(U, F, k) = 0 \tag{2}$$

where k is defined by

$$E(K, k, K) = 0 \text{ or } E(k, K, k) = 0. \tag{3}$$

Computationally it will pay to calculate this reciprocal kernel once and for all: (a) if many different problems are to be solved, involving a multitude of different F functions but always the same K; (b) if F is a function of x and z rather than of x alone, since in this case the conditions of (a) are realized; (c) if K is of the Poisson-Wiener type; (d) or if K happens to be of some special simple analytic form, yielding an easy recognizable solution.

Otherwise it will not usually pay to calculate the reciprocal kernel, just as it does not usually pay to calculate the inverse to a matrix if only one set of linear equations is to be inverted. This is because calculating or approximating the inverse of a matrix involves operations like multiplying a square matrix into a square matrix, which is n times as much work as to multiply a square matrix into a single vector of n components. Similarly, to compute a reciprocal kernel we must evaluate integrals of the form $A(x, z)\cdot B(x, z)$, which is of a higher dimensionality of work than to compute integrals of the form $A(x, z)\cdot B(x)$. However, as will be shown, any kind of an approximation to k can greatly accelerate the convergence to the true U solution.

3. A commonly used sequence of approximations to U is given by $U_0, U_1, \cdots, U_n$, where U_n satisfies the iteration

$$U_{n+1} = -F - K\cdot U_n = U_n - E(F, U_n, K), \qquad U_0 \text{ arbitrary.} \tag{4}$$

If we set $U_0 = 0$, and calculate U_n as $\sum_1^n (U_i - U_{i-1})$, where these last differences satisfy a relation like (4) but without the F, then we have the Liouville-Neumann series; hence, for simplicity we can call (4) a Neumann sequence.

Provided our original problem has a solution, this sequence will either converge for all U_0 or else will diverge for almost all initial U_0. It is sufficient for convergence that K be of Volterra type, vanishing for $z > x$. Also, it is sufficient for convergence, but not necessary, that K be sufficiently small in absolute value so that its "norm," $N(K)$, satisfies

$$N(K) = \max_x \int_a^b |K(x, s)|\, ds < 1, \tag{5}$$

where the so-defined norm can be shown to have the property

$$N(A\cdot B) \leqq N(A)N(B) \qquad \text{for all } A \text{ and } B. \tag{6}$$

Complete necessary and sufficient conditions for the convergence of (3) are that the eigenvalues of the associated homogeneous integral equation all satisfy the following inequality

$$|\sigma^{-1}| = |m| < 1 \text{ where } E(0, V, -\sigma K) = 0, \qquad V \not\equiv 0. \tag{7}$$

The speed of asymptotic convergence to U will depend upon the smallness of the largest modulus of m. Note that if the Neumann sequence is defined as in (4) and is convergent, then a numerical error in iteration will be later self-correcting.

4. In many actual applications there is no reason why the norm of K or the dominating eigenvalue should be such as to make the Neumann iteration converge. Even if convergence is assured, as in the Volterra case, there is no guarantee that the convergence will be at all rapid. The reason is not hard to find: from the last expression in (4) it is apparent that the Neumann sequence does utilize the error, E, that comes from substituting an approximate U_n in (1). But it utilizes that error in rather an arbitrary way; at any point (x, z) we simply subtract E from U_n to get a new approximation, and such a procedure in no way suggests itself as being optimal.

However, if we had knowledge of the reciprocal kernel, k, it is easy to see what would be the correct way to utilize E. The expression

$$U_{n+1} = U_n - E(F, U_n, K) - k \cdot E(F, U_n, K) \tag{8}$$

can by substitution be verified to give $U_{n+1} = U$ in one step regardless of U_n and regardless of the size of K. Hence, if we know any approximation to $k(x, z)$, which we may write as $j(x, z)$, then we should expect that using it in place of k in (8) will yield a modified Neumann sequence that is much more rapidly convergent than (4).

That this is rigorously the case can be seen by a different argument. Using any arbitrary function $j(x, z)$, which happens to be an approximation to $k(x, z)$ but need not be restricted to being so, we can add and subtract $j \cdot F$ to (1) and evaluate the resulting expression in terms of U; it turns out to give us a new integral equation for U

$$\begin{aligned} 0 &= (F + j \cdot F) + U + (j + K + j \cdot K) \cdot U \\ &= E(F + j \cdot F, U, j + K + j \cdot K). \end{aligned} \tag{9}$$

Now this is seen to be of exactly the same form as (1) but with a new known F and a new kernel

$$j + K + j \cdot K = E(j, K, j). \tag{10}$$

Comparing (10) with the second form of (3), we see that if j were identical to the reciprocal kernel, k, the new kernel would vanish and (9) could provide an explicit solution to our problem. To the degree that j is close to k and almost satisfies (3), the new kernel will be very small, and hence applying the Neumann sequence (4) to the new equation (10), we arrive at the modified Neumann sequence in any of the equivalent forms

$$\begin{aligned} U_{n+1} &= -F - j \cdot F - (j + K + j \cdot K) \cdot U_n \\ &= U_n - E(F + j \cdot F, U_n, j + K + j \cdot K) \\ &= U_n - E(F, U_n, K) - j \cdot E(F, U_n, K). \end{aligned} \tag{11}$$

Note that the last of these is precisely of the form (8) but with the j approximation to the reciprocal kernel used rather than k itself. Hence, the earlier argument of (8) for utilizing the error E has been confirmed.

The last expression in (11) is computationally the more convenient if F is a

function of x alone and if only a few iterations are needed. This for two reasons: integrals of the form $j(x, z) \cdot U_n(x)$ are much less labor to compute than an integral of the form $j(x, z) \cdot K(x, z)$; also, the error of the original equation (1), $E(F, U_n, K)$ may be of intrinsic physical interest for its own sake. On the other hand, if F is a function of z as well as x and if many iterations are going to be needed, computing the new kernel once and for all will save the need to later evaluate integrals of the form $j \cdot E(x, z)$. Also, the new transformed equation will presumably have so small a kernel that it will be much better conditioned than (1) itself, and its error will already provide a very good approximation to $U - U_n$.

It goes without saying that we ought not to pick j with the singular property of making (9) not have a solution, as would be the case if the new kernel had an eigenvalue equal to unity. On the contrary, we wish to pick j near in some sense to k. As (9) and (10) show, a criterion for nearness of j to k is that $E(j, K, j)$ be everywhere small in absolute value so that its norm will be small and so that the $| m |$ of (7), defined for the new kernel, will be small and the convergence to U rapid.

5. Whether (4) or (11) is used and whether or not either is rapidly or slowly convergent or even divergent, after a number of iterations U_n primarily differs from U by a term proportional to $(m_1)^n$ where m_1 is the dominant root defined by a homogeneous equation of type (7). It can be shown to follow that we can usually speed up convergence by computing at a late stage of the iteration

$$U \cong \frac{U_n U_{n+2} - U_{n+1}^2}{(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)}. \tag{12}$$

As far as I know, the exact conditions for convergence of this process and its speed of convergence have not been rigorously settled except in the case of a one-variable iteration, but there are intuitive reasons to expect this device to accelerate convergence. (See Aitken [1] and Samuelson [6].)

6. To apply the modified and generalized Neumann sequence (11), we need j, an approximation to K's reciprocal kernel, k; but we do not need to know j's own reciprocal, which might be called J, and which is an approximation to K itself. The way in which we get our j approximation is of no consequence for the above method. A good guess, or the similarity of this problem to another problem already solved, or the knowledge of the reciprocal kernel of K for some nearby parameter value, all these are possibilities. However, there is no reason why we cannot consider the problem of systematically improving on an approximation to the reciprocal kernel k itself, since it has been pointed out in (3) above that the problem of getting k is a special case of our general formulation—namely the case when F and K happen to be identical. Thus, our new modified sequence for k is given $k_0, k_1, k_2, \cdots, k_n, \cdots$, where

$$\begin{aligned} k_{n+1} &= k_n - E[K + j \cdot K, k_n, E(j, K, j)] \\ &= k_n - E(K, k_n, K) - j \cdot E(K, k_n, K), \end{aligned} \tag{11'}$$

where the initial k_0 is arbitrary and may or may not be set equal to zero so as to give k_1 equal to j itself. The convergence of this sequence has already been discussed in connection with (11).

7. Like all convergent linear iterations that are not exact, the above process approaches the true k solution so as to give ultimately a number of additional correct decimal places proportional to the number of additional iterations. For many practical purposes this is good enough, especially if the accelerating device (12) is used after we have performed a number of linear iterations.

Nonetheless, a way immediately suggests itself to greatly speed up the convergence of the modified Neumann sequence (11)′. It will be recalled that its speed of convergence is conditioned upon our using for j the best known approximation to the true reciprocal kernel, k. But note that k_n in (11)′ is presumably a much closer approximation to k than k_0 or j. So it is natural to replace j in (11) by k_n itself, leading to the desired new sequence

$$k_{n+1} = k_n - E(K, k_n, K) - k_n \cdot E(K, k_n, K). \tag{13}$$

This is no longer a linear iteration; i.e., k_{n+1} is a quadratic rather than a linear functional of k_n. Its convergence in a close neighborhood of k turns out to be more rapid than that of any linear iteration. This can be verified by an extension of the Liapunov stability theory to functionals: expanding the non-linear functional in a Taylor-like series in terms of functional derivatives around the true solution $k(x, z)$, we find that all linear terms vanish. From this it follows that asymptotically the number of correct decimal places grows in geometric progression with the number of iterations: i.e., $\log | k_n - k |$, for any x and z, ultimately diminishes like -2^n. This fact can be verified directly if we manipulate (13) into a verifiably equivalent form

$$E(K, k_{n+1}, K) = -E(K, k_n, K) \cdot E(K, k_n, K) \tag{13′}$$

with

$$N[E(K, k_{n+1}, ,K)] \leqq N[E(K, k_n, K)]^2$$

because of the property of (6). This means that the ultimate error is each time being squared, so that if we start out with a small enough error, convergence will be assured. (For the corresponding non-linear Newton-like iteration applied to inverting a finite matrix, see Frazer, Duncan, Collar [3]; by taking the limit of a large enough number of linear equations, the Fredholm integral equations can be shown to have all the same properties so that heuristically we may invent new methods and find proofs for them by extending to integral equations the various proposals made for matrices.)

The formulation (13′) enables us to give a sufficient condition for the neighborhood of convergence. For all k_0 with $N[E(K, k_0, K)] < 1$, convergence will be assured. This is of course not a necessary condition: a sequence that is going to converge may begin with a norm greater than unity; and for a transient period, its norm may actually grow rather than diminish.

One might at first think that a necessary and sufficient condition like that relating to the eigenvalue of (7) could be found, but it turns out that the problem of delineating the exact *region* of convergence of a non-linear iteration like that of (13) is theoretically a very delicate task. It has the pleasing quality of necessarily being locally convergent for all K provided we have a sufficiently close k_0 to k; moreover, the ultimate rate of local convergence is exceedingly rapid, as we have seen. But there has sometimes been observed a tendency for those iterations that are most powerful locally to have to pay the price of having a restricted domain in the large from which they will converge.

A little light on this problem of region of convergence is provided by recognizing that (13) is essentially a generalization to functions of infinitely many variables of the familiar Newton iteration, by which we find the root of an implicit equation by extrapolating along its tangent line at some approximate root and thereby inferring a better approximation where the tangent line intersects with the zero axis. Such an iteration is known to give rapid convergence of the second order in the neighborhood of any simple root.

To bring out the Newton analogy with the present problem, imagine that the implicit equation for k given by (3) referred to ordinary algebraic variables K and k, and that the integral of products could be interpreted as a simple product. This is not so far-fetched as might be the case, since in the special case where K happens to be independent of x and z, our problem becomes precisely of this type. We can write (3)

$$(3') \qquad 0 = E(K, k, K) = K + k + Kk = (1 + K)(1 + k) - 1,$$

or

$$0 = [1/(1 + k)] - (1 + K) = g(k), \qquad K \neq -1 \neq k.$$

Now applying the usual Newton-Raphson iteration (Whittaker and Robinson [9]), we have

$$\begin{aligned} k_{n+1} &= k_n - \frac{g(k_n)}{g'(k_n)} = k_n - \frac{[1/(1 + k_n)] - (1 + K)}{[-1/(1 + k_n)^2]} \\ &= k_n + (1 + k_n) - (1 + k_n)\{(1 + k_n)(1 + K)\} \\ &= k_n - E(K, k_n, K) - k_n E(K, k_n, K), \end{aligned} \qquad (13'')$$

which is seen to be of the same form as (13). Since (3′) and (13″) refer to simple hyperbolas, it is easy to determine conditions for the neighborhood of convergence. In this simple case it turns out that the modified Neumann process (with $j = k_1$) and the Newton processes have exactly the same region of convergence, namely $| K + k_0 + Kk_0 | < 1$. The reason for this special coincidence is the fact that in a one-variable problem, the norm sufficiency condition is also necessary: the norm must change in a monotonic way with each iteration, which is not at all true in the general case.

In practical applications there may be knowledge available that helps us to find an approximation (to the reciprocal kernel) which will serve as a satisfactory

j to insert in (11) or which can be used in (13) to get a more adequate approximation or a self-improving sequence of functions. It is just as well, however, to indicate how in theory we can always construct as close an approximation to k as we wish, even when we know little or nothing that helps us identify regions of convergence. Replace K by λK and consider moving from $\lambda = 0$ to $\lambda = 1$, which gives us our original problem. Now we know that for sufficiently small values of λ, the simple Neumann process will converge; hence we can calculate reciprocal kernels to λK, which we call k_λ, for all small λ. Moreover, it is known from the Fredholm theory that the eigenvalues of λK are isolated: therefore, in the complex plane for λ, there are only a finite number of eigenvalues in any finite circle including the origin. We can always construct a continuous path from 0 to 1 that avoids all such critical points. If we take sufficiently small steps along this path, we can certainly by the modified Neumann process, always find a convergent series for k_λ at every step; and proceeding in this way, we can finally find a convergent series for k itself. This is of course roundabout and would presumably be used rarely in practice, but it can always be done rigorously.

8. In conclusion, we can utilize the formulation (11) to classify a number of suggested methods of approximation in terms of what it is that they assume about K and j.

Case 1. If we put $j = 0$, we get the simple Neumann series (4). For this to be useful, the reciprocal kernel k must be almost zero, which obviously means that K itself must be almost zero. This checks with our derived stability conditions for convergence of the ordinary Neumann sequence.

Case 2. If we put j equal to a constant independently of x and z, we get the method attributed by Wagner [7] to Wiarda [9] and Bückner [2]. For this method to be optimal, k and K must not be too strongly dependent on x and z, as in the discussion of the Newton-Raphson problem. The average values $\bar{K} = \int_a^b \int_a^b K(x, z)\, dz\, dx/(b-a)^2$ and $\bar{j} = -\bar{K}/(1+\bar{K})$ then recommend themselves for J and j, though to play it safe, convergence will be assured for sufficiently small $\bar{j}$.

Case 3. If K, and k, are not too strongly dependent on z so that $K(x, z) \cong \bar{K}(x) = \int_a^b K(x, s)\, ds/(b-a)$, then we can easily solve for the reciprocal kernel to $\bar{K}(x)$ and set $j(x) = -\bar{K}(x) \Big/ \left(1 + \int_a^b \bar{K}(s)\, ds\right)$. Putting this in the modified Neumann sequence (11) gives us almost the same method as that recommended by Wagner [8]. However, as Dr. Wagner has kindly pointed out to me, there are some minor differences: for one thing, he found it most convenient for his problem to apply this procedure after several ordinary Neumann iterations had been performed, instead of from the beginning; also, in equation (8) of his paper, he uses the average value of his $\Delta^{(n)}(x)$, whereas I use its actual value at each x.

Case 4. If K and k are not too dependent on x, we can simply transpose the roles of x and z in Case 3 and define $j(z)$ in a corresponding fashion. The corresponding modified Neumann sequence can be defined easily.

Case 5. If we approximate K and k by the simple product of a function of x and a function of z, we get all of the previous cases as special cases. Thus, use the following estimates

$$K(x, z) \cong J(x, z) = A(x)B(z)$$

$$A(x) = \int_a^b K(x, s)\, ds, \qquad B(z) = \int_a^b K(s, z)\, ds \Big/ \int_a^b \int_a^b K(x, z)\, dx\, dz. \tag{14}$$

By substitution or by consulting any text on integral equations, we can easily verify that the reciprocal kernel to a J of this form must be

$$j(x, z) = -qA(x)B(z) \quad \text{where} \quad q = \left(1 + \int_a^b A(s)B(s)\, ds\right)^{-1}. \tag{15}$$

Hence, we can easily define our modified Neumann sequence, and at each iteration this will involve only one quadrature of E rather than a new quadrature at each x. This Case 5 is clearly a generalization of all the previous cases, which emerge when A and B are given suitable constant values. As mentioned below, a further generalization would be possible in which J is written as the sum of a number of terms which are each products of a function of x and a function of z; the reciprocal kernel j can then be determined merely by solving a finite set of linear algebraic relations, whose coefficients are given by single quadratures. At the end of this paper, a numerical example is given to illustrate the general principles involved in the Neumann sequence and in Case 5.

It is to be noted that, even when K cannot be satisfactorily approximated by a product of functions of each variable, our problem can sometimes be converted into a new form in which this approximation is valid. Thus, $E(F + K \cdot F, U, K \cdot K) = 0$ can be easily verified to be equivalent to our original problem $E(F, U, K) = 0$. The new kernel $K \cdot K$ may be more capable of being closely approximated by a simple product; and should the approximation still not be adequate, we can convert our new problem into still a newer problem in exactly the same way just indicated. Provided all the eigenvalues of K exceed unity, except possibly for one simple real root, we can be sure that after a finite number of transformations our approximation will become adequate. Since integrals of the form $K \cdot K$ are laborious to compute, these transformations are not always to be recommended.

Case 6. If K has a very high peak along $x = z$, so too will k. If $K(x, z)$ can be approximated by $K(x)\delta(x - z)$, where δ refers to the Dirac impulse function with the property $\int_a^b K(x, s)\, ds = \int_a^b K(x)\delta(x - s)\, ds = K(x)$, then we set $j(x, z) = (-1 + c(x))\delta(x - z)$, where $c(x) = -1(1 + K(x))^{-1}$. This represents the interesting method recently proposed by Wagner [7]. Applying our norm condition to $E(j, K, j)$ in the usual fashion gives Wagner's sufficiency condition for convergence. It may be possible to extend this method by utilizing Dirac functions that depend upon terms like $x - g(z)$.

Case 7. Especially if K is symmetrical and positive definite, it may pay to approximate it by a function $J(x, z) = K(x, z)$ for $z \leqq x$, but which vanishes

elsewhere. This approximation will then be of Volterra type, and even its simple Neumann series will be everywhere convergent so that its reciprocal kernel can be evaluated in terms of iterated integrals. Calling this reciprocal kernel j, we can proceed with the modified Neumann series; and from the known theory of similar iterations of definite matrixes, (von Mises and Pollaczek-Geiringer [4], Morris [5]) we can be sure the resulting sequence (11) will necessarily be convergent. Moreover, we can avoid calculating integrals of the type $J(x, z) \cdot A(x, z)$ involved in the calculation of the reciprocal kernel j, if we write and solve (11) in the verifiably equivalent form

$$\{F + (K - J)\cdot U_n\} + U_{n+1} + J\cdot U_{n+1} = 0, \qquad n = 0, 1, \cdots . \tag{16}$$

Since we know that a Volterra J leads to a convergent ordinary Neumann series, we can in early iterations content ourselves with a few Neumann whirls of type (4), never calculating any integrals worse than of the form $J(x, z)\cdot A(x)$. Later we can strive for more accuracy. Or if we like, we can rather rapidly get approximate solutions to (16) by means of a net of approximating linear equations, whose matrix will be triangular and hence easily solved. Even if K is not symmetric, if it is everywhere negative and such as to make (4) convergent, then from the theory of Frobenius non-negative matrices, we can show that (16) is even more rapidly convergent.

Even if K is not symmetric or definite, we can always convert our problem into a new problem with this form. We calculate $\phi = \int_a^b E(F, U, K)^2\, dx$ and seek U so as to minimize ϕ, leading to a new integral equation for U of the form (1) but with $F(x, z) + \int_a^b K(s, x)F(x, z)\, ds$ substituted for F, and the symmetric kernel $K(x, z) + K(z, x) + \int_a^b K(s, x)K(s, z)\, ds$ substituted for K. Note that it is $\delta(x - z)$ + the new kernel that is positive definite, and not necessarily the new kernel alone.

Case 8. Any Fredholm equations can be solved by using a finer and finer grid of linear approximations. We can think of the relevant rectangle of the (x, z) plane as being covered by a lattice of equally distant points, at each of which we evaluate K, and then proceed to set up and solve ordinary algebraic equations. This approximation to K can be written in terms of suitably defined Dirac functions and called J; its inverse, j, will also consist of certain defined Dirac functions, whose coefficients emerge from the solution of ordinary linear equations. Provided our norm condition is satisfied for j, we may remain content in most of the iterations with a rather coarse grid, going through more refined calculations with a finer grid only when it comes to evaluating the integrals in the expression $E(F, U_n, K)$, and only bothering with such refinement at a late stage in the iterations when the accuracy has some meaning.

Case 9. Needless to say, to the extent that we can interpolate for K by any one of the usual methods—such as polynomial interpolation, sums of exponentials, sums of products of functions of the form $\sum f_k(x)g_k(z)$, etc.—than a suitable

J and j can be defined and inserted into (11). The important thing is to utilize the information, which comes to us from simply checking how well our equation is satisfied, so as to best improve our estimate of the true solution.

9. A numerical example may be used to illustrate the general principles involved in the above discussion and also the special methods suggested in Case 5. Consider

$$U(x) = 1.5x + 1.5 - \int_0^1 (x + xs + s)U(s)\,ds \quad \text{or} \tag{16}$$

$$E(-1.5x - 1.5, U, x + xz + z) = 0$$

whose solution can be verified to be $U(x) = 1$. The first few Neumann iterations give $U_0(x) = 0$, $U_1(x) = 1.5x + 1.5$, $U_2(x) = -2x + .25 \cdots$, and this sequence can be easily shown to diverge in antidamped oscillations. (Actually the simple Neumann sequence with $C_nx + D_n$ fed into it at any stage gives back a similar linear expression but with new C_{n+1} and D_{n+1} coefficients. These new coefficients can be defined in terms of the old by simple linear relations:

$$C_{n+1} = 1.5 - \tfrac{5}{6}C_n - \tfrac{3}{2}D_n, \qquad D_{n+1} = 1.5 - \tfrac{1}{3}C_n - \tfrac{1}{2}D_n,$$

which can be verified to have a fixed point at $(C, D) = (0, 1)$. But by examining its 2 x 2 matrix, we find a characteristic root $\lambda = -(4 + \sqrt{19})/6$, which exceeds 1 in absolute value, so that the fixed point is unstable and even in its close vicinity we diverge away from it.)

We can easily apply the method of Case 5 to the modified Neumann sequence (11) with better results. For the case, $K(x, z) = x + xz + z$, we evaluate equations (14) and (15) as follows:

$$K = x + xz + z \cong J = A(x)B(z) = \tfrac{1}{3}(3x + 1)(3z + 1), \quad \text{where}$$

$$A(x) = \int_0^1 (x + xs + s)\,ds, \text{ etc.}$$

$$j(x, z) = -\tfrac{1}{12}(3x + 1)(3z + 1) \quad \text{with} \quad 12^{-1} = 1 + \int_0^1 A(s)B(s)\,ds.$$

The modified Neumann sequence (11) can now be written in the following form:

$$E_n(x) = -1.5x - 1.5 + U_n(x) + \int_0^1 (x + xs + s)U_n(s)\,ds$$

$$U_{n+1}(x) = U_n(x) - E_n(x) + \tfrac{1}{12}(3x + 1)\int_0^1 (3s + 1)E_n(s)\,ds.$$

The first few iterations yield the following $U_1 = 1.5x + 1.5$, $U_2 = \tfrac{3}{32}x + (1 - \tfrac{5}{96})$, $U_3 = \tfrac{3}{512}x + (1 - \tfrac{5}{1536})$, $\cdots$, which is seen to be rapidly convergent. Actually, if we had started with $U_0 = 0$, we should have gotten $U_1 = 1$, the exact answer, in a single step. This is because of the quite accidental

fact that the Modified Neumann sequence gives us for $U_n = C_n x + D_n$ the *singular* linear iteration

$$C_{n+1} = 0 + \tfrac{1}{16}C_n + 0D_n\,, \qquad D_{n+1} = 1 - \tfrac{5}{144}C_n + 0D_n\,,$$

which is seen to have a *zero* characteristic root, and also a heavily damped $\lambda = \tfrac{1}{16}$. Another way of verifying convergence is to calculate (10)

$$\begin{aligned} E(K, j, K) &= x + xz + z - \tfrac{1}{12}(3x + 1)(3z + 1) \\ &\qquad - \tfrac{1}{12}(3z + 1)\int_0^1 (x + xs + s)(3s + 1)\,ds \\ &= \tfrac{1}{24}(10x - 18xz + 9z - 5), \end{aligned}$$

which, in absolute value never exceeds $\tfrac{5}{24}$ and whose norm is even less. For K such a simple polynomial in x and z, we can easily get the exact reciprocal kernel by trying a similar polynomial expression whose coefficients can be determined by substitution into (3) to be

$$k = \tfrac{4}{27} - \tfrac{2}{3}x - \tfrac{2}{3}z + 0xz,$$

so that the error in the j used in our Modified Neumann series is verified to be quite small and never to exceed $\tfrac{8}{27}$ at any point.

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AN IDENTITY SATISFIED BY THE RACAH COEFFICIENTS*

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1. In discussing the theory of complex spectra[1], Racah found it expedient to introduce the function, subsequently called the Racah coefficient,

$$W(abcd;ef) = \Delta(abe)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf)$$
$$\cdot \sum_z (-1)^z \frac{(a+b+c+d+1-z)!}{(a+b-e-z)!\,(c+d-e-z)!\,(a+c-f-z)!}$$
$$\cdot \frac{1}{(b+d-f-z)!\,z!\,(e+f-a-d+z)!\,(e+f-b-c+z)!}$$
$$\Delta(abe) \equiv \left[\frac{(a+b-e)!\,(a-b+e)!\,(-a+b+e)!}{(a+b+e+1)!}\right]^{\frac{1}{2}} \qquad (1)$$

In this sum z takes on all integer values, including zero. W is defined for all integer and half integer values of its arguments subject to the restriction that each of the triads (abe), (cde), (acf), (bdf) has an integer sum. The arguments have the significance of angular momentum vectors and the Racah coefficient can be intepreted as the transformation coefficient[2] from the coupling scheme $(\mathbf{a}+\mathbf{b}=\mathbf{e},\ \mathbf{e}+\mathbf{d}=\mathbf{c})$ to the scheme $(\mathbf{b}+\mathbf{d}=\mathbf{f},\ \mathbf{a}+\mathbf{f}=\mathbf{c})$. This point will be discussed more fully below, along with other properties of the coefficients, using the procedure of ref. 2.

Recently it has been found that the Racah coefficients are useful in discussing many other topics of current physical interest: the theory of the angular correlation of successive radiations[3-7], the angular dependence of scattering and reaction cross sections[8], nuclear structure[9], to mention a few. A need for more explicit algebraic formulae for the W coefficients (specifying values for one or more of the variables) as well as numerical tabulation became apparent. Jahn[9] recently published tables of algebraic formulae for the W's similar in form to the well-known tabulation of the vector addition coefficients[10]. The numerical tabulation was undertaken by Blatt and the author[11], in connection with a review paper on

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the angular dependence of nuclear and reaction cross sections. In setting up the task of numerical calculation it became clear that neither (1) nor the more explicit algebraic forms deduced from it[9] was the best way to proceed. A more desirable procedure would involve computation from recursion formulae, that should exist, of course. In developing such recursion formulae, a quite general identity for the Racah coefficients was found, which is the subject of this note. The significance of this identity for the theory of the W coefficients has been discussed by Racah[12], and this note will therefore be confined to a simple demonstration of the identity only.

Besides their application to problems of physical interest, the Racah coefficients are of considerable mathematical interest for group theoretical problems. In particular, Racah has discussed the application to an algebra of tensor operators and the construction of matrices that decompose the Kronecker product.[13]

2. Let us consider normalized wave functions of the form $\psi(jm)$ which are eigenfunctions of the operators $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z with eigenvalues $j(j+1)$ and m respectively. J is here the total angular momentum operator, and satisfies the commutation relations $[J_x,J_y] = iJ_z$ etc. If we have two such wave functions, for different particles, we can form a composite wave function that is the simultaneous eigenfunction of the operators $\mathbf{J}_1^2$; $\mathbf{J}_2^2$; $(\mathbf{J}_1 + \mathbf{J}_2)^2 = \mathbf{J}^2$; $J_z = J_{1z} + J_{2z}$ in the usual way using the vector addition coefficients[10]:

$$\Phi(JM) = \sum_{m_1} (j_1 j_2 m_1 M - m_1 \mid j_1 j_2 JM)\psi(j_1 m_1)\psi(j_2 M - m_1) \qquad (2)$$

The vector addition coefficient $(ab\alpha\beta \mid ab c\gamma)$ vanishes unless $\gamma = \alpha + \beta$. In (2)' and in subsequent equations, we have explicitly satisfied this requirement, eliminating a formal sum over γ. The vector addition coefficients are unitary (and real), satisfying the relations[10]

$$\sum_\alpha (ab\alpha\gamma - \alpha \mid abc\gamma)(ab\alpha\gamma - \alpha \mid abc'\gamma) = \delta_{cc'}$$
$$\sum_c (ab\alpha\gamma - \alpha \mid abc\gamma)(ab\alpha'\gamma - \alpha' \mid abc\gamma) = \delta_{\alpha\alpha'} .$$

It follows from this, and the orthonormality of the $\psi(j_i m_i)$, that the $\Phi(JM)$ are also orthonormal.

If we now consider three particles, we can define composite wave functions of sharp total angular momentum, $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3$, in a similar way. The composite wave functions so defined are, however, not unique but depend upon the coupling scheme. If, for example, we couple first j_1 and j_2 to form J_a and then couple J_a and j_3 to get J we get the composite wave functions:

$$\Phi(j_1 j_2 (J_a) j_3 \,;\, JM)$$
$$= \sum_m [\sum_{m_1} (j_1 j_2 m_1 m - m_1 \mid j_1 j_2 J_a m)\psi(j_1 m_1)\psi(j_2 m - m_1)] \qquad (3)$$
$$\cdot (J_a j_3 mM - m \mid J_a j_3 JM)\psi(j_3 M - m)$$

[12] G. Racah, (unpublished). Dr. U. Fano kindly let us see his copy of a preliminary form of the manuscript.

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Here the good quantum numbers for the composite wave function are J, M, J_a, and, of course, j_1, j_2, j_3. The notation for Φ is chosen to indicate the coupling scheme. On the other hand we might have coupled first j_2 and j_3 to form J_b and then coupled j_1 and J_b to form J. This yields the wave function:

$$\Phi(j_1, j_2 j_3(J_b); JM) = \sum_m [\sum_{m_2} (j_2 j_3 m_2 m - m_2 \mid j_2 j_3 J_b' m) \psi(j_2 m_2) \psi(j_3 m - m_2)] \cdot (j_1 J_b M - mm \mid j_1 J_b J M) \psi(j_1 M - m) \tag{4}$$

The two sets of wave functions defined by (3) and (4) are easily shown to be orthonormal, that is to say:

$$\langle \Phi(j_1 j_2(J_a') j_3 ; J'M') \mid \Phi(j_1 j_2(J_a) j_3 ; JM) \rangle = \delta_{JJ'} \delta_{MM'} \delta_{J_a J_a'} \tag{5}$$

Similarly for the wave functions (4).

Since both these sets of wave functions span the same space, it is clear they must be related by equations of the form*:

$$\Phi(j_1, j_2 j_3(J_b); JM) = \sum_{J_a} \langle j_1, j_2 j_3(J_b) J \mid j_1 j_2(J_a) j_3 J \rangle \Phi(j_1 j_2(J_a) j_3 ; JM) \tag{6}$$

Using the orthonormality of the two sets of wave functions we deduce that the transformation is unitary:

$$\sum_{J_a} \langle j_1, j_2 j_3(J_b) J \mid j_1 j_2(J_a) j_3 J \rangle^* \langle j_1, j_2 j_3(J_b') J \mid j_1 j_2(J_a) j_3 J \rangle = \delta_{J_b J_b'} \tag{7}$$

Furthermore, by substituting (3) and (4) into (6) using the orthonormality of the $\psi(j_i m_i)$ we find:

$$(j_2 j_3 m_2 m - m_2 \mid j_2 j_3 J_b m)(j_1 J_b M - mm \mid j_1 J_b J M) = \sum_{J_a} \langle j_1, j_2 j_3(J_b) J \mid j_1 j_2(J_a) j_3 J \rangle \cdot (j_1 j_2 M - m m_2 \mid j_1 j_2 J_a M - m + m_2)(J_a j_3 M + m_2 - mm - m_2 \mid J_a j_3 J M) \tag{8}$$

The unitarity of the vector addition coefficients then allows one to write:

$$\langle j_1, j_2 j_3(J_b) J \mid j_1 j_2(J_a) j_3 J \rangle = \sum_{m_1, m_2} (j_1 j_2 m_1 m_2 \mid j_1 j_2 J_a m_1 + m_2) \cdot (j_2 j_3 m_2 M - m_1 - m_2 \mid j_2 j_3 J_b M - m_1)(j_1 J_b m_1 M - m_1 \mid j_1 J_b J M) \cdot (J_a j_3 m_1 + m_2 M - m_1 - m_2 \mid J_a j_3 J M) \tag{9}$$

These last relations show incidentally that the transformation coefficients are real. Comparing (9) with Racah's work[1, 2] one finds:

$$\langle j_1, j_2 j_3(J_b) J \mid j_1 j_2(J_a) j_3 J \rangle = [(2J_a + 1)(2J_b + 1)]^{\frac{1}{2}} \cdot W(j_1 j_2 J j_3 ; J_a J_b) \tag{10}$$

* In (6) we used the fact that the transformation coefficient is independent of M. This can be seen by using the Wigner-Eckart theorem, Wigner, l. c. p. 264.

By using the properties of the vector addition coefficients* and (9), we find that the W coefficients have the symmetries:

$$\begin{aligned} W(abcd; ef) &= W(badc; ef) = W(cdab; ef) = W(acbd; fe) \\ &= (-1)^{a+d-e-f} W(ebcf; ad) = (-1)^{b+c-e-f} W(aefd; bc) \end{aligned} \tag{11}$$

From index balance we can deduce in (9) that the sum is independent of M (M is, of course, restricted to lie between $-J$ and J). Using this latter fact, Racah was able to perform the impressive algebraic feat of summing the right hand side of (9) to get (1).

Using (10) to rewrite (7) we find:

$$\sum_{J_a} (2J_a + 1)(2J_b + 1) W(j_1 j_2 J j_3; J_a J_b) W(j_1 j_2 J j_3; J_a J'_b) = \delta_{J_b J'_b} \tag{12}$$

In this notation (6) reads:

$$\begin{aligned} &\Phi(j_1, j_2 j_3 (J_b); JM) \\ &\quad = \sum_{J_a} [(2J_a + 1)(2J_b + 1)]^{\frac{1}{2}} W(j_1 j_2 J j_3; J_a J_b) \Phi(j_1 j_2 (J_a) j_3; JM) \end{aligned} \tag{6'}$$

Interchange now j_1 and j_2.

$$\begin{aligned} &\Phi(j_2, j_1 j_3 (J_b); JM) \\ &\quad = \sum_{J_a} [(2J_a + 1)(2J_b + 1)]^{\frac{1}{2}} W(j_2 j_1 J j_3; J_a J_b) \Phi(j_2 j_1 (J_a) j_3; JM) \end{aligned} \tag{13}$$

$$= \sum_{J_a} [(2J_a + 1)(2J_b + 1)]^{\frac{1}{2}} W(j_2 j_1 J j_3; J_a J_b) (-1)^{j_1 + j_2 - J_a} \Phi(j_1 j_2 (J_a) j_3; JM)$$

In getting (13) we used (3) and the fact that:

$$(j_1 j_2 m_1 m - m_1 \mid j_1 j_2 J_a m) = (-1)^{j_1 + j_2 - J_a} (j_2 j_1 m - m_1 m_1 \mid j_2 j_1 J_a m \tag{14}$$

Taking the inner product of (6′) and (13) yields:

$$\begin{aligned} &\langle \Phi(j_1, j_2 j_3 (J_b); JM) \mid \Phi(j_2, j_1 j_3 (J'_b); JM) \rangle \\ &\quad = \sum_{J_a} (2J_a + 1)[(2J_b + 1)(2J'_b + 1)]^{\frac{1}{2}} \cdot (-1)^{j_1 + j_2 - J_a} W(j_1 j_2 J j_3; J_a J_b) \\ &\qquad \cdot W(j_2 j_1 J j_3; J_a J'_b) \end{aligned} \tag{15}$$

This inner product, however, is itself a Racah coefficient and using (9) and (10), we find another relation[1] for the W's:

$$\begin{aligned} &W(j_1 J_b J'_b j_3; J j_2) \\ &\quad = \sum_{J_a} (2J_a + 1)(-1)^{j_1 + j_3 - J_a} W(j_1 j_2 J j_3; J_a J_b) \cdot W(j_2 j_1 J j_3; J_a J'_b) \end{aligned} \tag{16}$$

To proceed further it is a natural generalization to consider the coupling of more than three systems. For four wave functions, $\psi(j_i m_i)$, the composite func-

* These are the symmetry properties of the vector addition coefficients:

$$\begin{aligned} (ab a\beta \mid abc\gamma) &= (-1)^{a+b-c} (ba\beta a \mid bac\gamma) \\ &= (-1)^{a+b-c} (ab - \alpha - \beta \mid abc - \gamma) \\ &= (-1)^{a-\alpha} [(2c + 1)/(2a + 1)]^{\frac{1}{2}} (ac\alpha - \gamma \mid acb - \beta) \end{aligned}$$

and all other symmetries resulting from combinations of these basic symmetries.

tions of sharp J, M will involve two intermediate coupling vectors, as, for example, in the coupling scheme ($\mathbf{j}_1 + \mathbf{j}_2 = \mathbf{J}_a$, $\mathbf{j}_3 + \mathbf{j}_4 = \mathbf{J}_b$, $\mathbf{J}_a + \mathbf{J}_b = \mathbf{J}$) symbolized by $(j_1j_2(J_a), j_3j_4(J_b), J)$. There are 15 different types of coupling that are possible and, just as before, there exist unitary (real) transformations between the composite wave functions defined by each coupling scheme. One can write out the explicit form of these transformations in terms of six vector addition coefficients, in complete analogy to (9). Such transformations are naturally more general than the Racah coefficients, and are not without their own interest—a familiar example is the passage from (LS) to (jj) coupling for two electron wave functions. Fano[6] has shown that they are the transformations that occur in the general problem of the coupling of tensor parameters, or, as it applies to the particular problem of angular correlations, the correlation of successive radiations from a non-random initial state. The triple correlation problem[5] is another example of the occurrence of such transformations.

As one might expect, however, these more general transformations are expressible in terms of the Racah coefficients. If the transformation is between the coupling schemes $(j_1j_2(J_a), j_3j_4(J_b), J)$ to $(j_1j_2(J_a), j_3(J_d), j_4, J)$, or more generally between coupling schemes where only one of the intermediate couplings is unchanged, it is clear that only a single W coefficient is involved.

Consider next the transformation between the wave functions defined by the coupling scheme $(j_1j_2(J_a), j_3j_4(J_b), J)$ and the coupling scheme ($\mathbf{j}_2 + \mathbf{j}_3 = \mathbf{J}_c$, $\mathbf{j}_1 + \mathbf{J}_c = \mathbf{J}_d$, $\mathbf{J}_d + \mathbf{j}_4 = \mathbf{J}$) symbolized by $(j_1, j_2j_3(J_c), (J_d), j_4, J)$, that is:

$$\Phi(j_1, j_2j_3(J_c), (J_d)j_4; JM) = \sum_{J_aJ_b} \langle j_1, j_2j_3(J_c), (J_d), j_4, J \mid j_1j_2(J_a), j_3j_4(J_b), J\rangle \cdot\Phi(j_1j_2(J_a), j_3j_4(J_b); JM) \tag{17}$$

We can accomplish this transformation in two steps, each one of which involves a re-coupling of the type given by (5). First we change the coupling of j_1, j_2 and j_3 in the wave function on the left hand side of (17):

$$\Phi(j_1, j_2j_3(J_c), (J_d)j_4; JM) = \sum_{J_a} \langle j_1, j_2j_3(J_c)J_d \mid j_1j_2(J_a)j_3J_d\rangle\Phi(j_1j_2(J_a), j_3(J_d), j_4; JM) \tag{18}$$

and secondly re-couple J_aj_3 and j_4 in the wave functions appearing on the right hand side of (18) using (5) once again:

$$\Phi(j_1j_2(J_a), j_3(J_d), j_4; JM) = \sum_{J_b} \langle J_aj_3(J_d)j_4J \mid J_a, j_3j_4(J_b)J\rangle\cdot\Phi(j_1j_2(J_a), j_3j_4(J_b); JM) \tag{19}$$

Substituting (18) and (19) in (17) yields the result:

$$\langle j_1, j_2j_3(J_c), (J_d), j_4J \mid j_1j_2(J_a), j_3j_4(J_b), J\rangle = \langle j_1, j_2j_3(J_c)J_d \mid j_1j_2(J_a)j_3J_d\rangle\cdot\langle J_aj_3(J_d), j_4J \mid J_a, j_3j_4(J_b)J\rangle \tag{20}$$

This procedure is, however, not unique. We might equally well have applied (5) three times in the sequence:

(a) Recouple j_1, J_c and j_4 in the wave function on the left hand side of (17):

$$\Phi(j_1, j_2 j_3(J_c), (J_d), j_4; JM) = \sum_\alpha \langle j_1 J_c(J_d) j_4 J \mid j_1, J_c j_4(\alpha) J\rangle \Phi(j_1; j_2 j_3(J_c), j_4(\alpha); JM) \tag{21}$$

(b) Recouple $j_2 j_3$ and j_4 in the wave functions on the right hand side of (21):

$$\Phi(j_1; j_2 j_3(J_c) j_4(\alpha); JM) = \sum_{J_b} \langle j_2 j_3(J_c) j_4 \alpha \mid j_2, j_3 j_4(J_b)\alpha\rangle \Phi(j_1; j_2, j_3 j_4(J_b), \alpha; JM) \tag{22}$$

(c) Finally, recouple $j_1 j_2$ and J_b in the wave functions on the right hand side of (22):

$$\Phi(j_1; j_2, j_3 j_4(J_b), \alpha; JM) = \sum_{J_a} \langle j_1, j_2 J_b(\alpha) J \mid j_1 j_2(J_a) J_b J\rangle \Phi(j_1 j_2(J_a), j_3 j_4(J_b); JM) \tag{23}$$

Putting these steps together we now obtain:

$$\begin{aligned}\langle j_1, j_2 j_3(J_c), (J_d), j_4 J \mid j_1 j_2(J_a), j_3 j_4(J_b), J\rangle &= \sum_\alpha \langle j_1 J_c(J_d) j_4 J \mid j_1, J_c j_4(\alpha) J\rangle \cdot \langle j_2 j_3(J_c) j_4 \alpha \mid j_2, j_3 j_4(J_b)\alpha\rangle \\ &\quad \cdot \langle j_1, j_2 J_b(\alpha) J \mid j_1 j_2(J_a) J_b J\rangle\end{aligned} \tag{24}$$

By comparing (20) and (24), we find the desired identity for the Racah coefficients. In the notation of (10), and written more symmetrically by means of (11), the identity reads:

$$W(a\alpha b\beta; c\gamma) W(\bar{a}\alpha\bar{b}\beta; \bar{c}\gamma) = \sum_\lambda (2\lambda + 1) W(\bar{a}\lambda\alpha c; a\bar{c}) W(b\lambda\beta\bar{c}; \bar{b}c) W(\bar{a}\lambda\gamma b; a\bar{b}) \tag{25}$$

Since our interest is primarily in exhibiting this result, suffice it to say that all the remaining transformations between coupling schemes for four wave functions can be expressed as sums of products of Racah functions.

It has been pointed out by Racah[12] that, aside from an indeterminate phase, (11) (12) (16) and (25) determine the W coefficients completely. It follows from this important result that further relations independent of these do not exist.

3. The application of the identity (25) to give a recursion relation is immediate. Take, for example, $\bar{c} = \frac{1}{2}$. Then in the sum λ takes on but two values $\lambda = c \pm \frac{1}{2}$. Solving for the term with $\lambda = c + \frac{1}{2}$ yields:

$$\begin{aligned}W(\bar{a}c + \tfrac{1}{2}\gamma b; a\bar{b}) \cdot [2(c+1) W(\bar{a}c + \tfrac{1}{2}\alpha c; a\tfrac{1}{2}) W(bc + \tfrac{1}{2}\beta\tfrac{1}{2}; \bar{b}c)] \\ = W(a\alpha b\beta; c\gamma) \cdot [W(\bar{a}\alpha\bar{b}\beta; \tfrac{1}{2}\gamma)] - W(\bar{a}c - \tfrac{1}{2}\gamma b; a\bar{b}) \\ \cdot [2c W(\bar{a}c - \tfrac{1}{2}\alpha c; a\tfrac{1}{2}) W(bc - \tfrac{1}{2}\beta\tfrac{1}{2}; \bar{b}c)]\end{aligned} \tag{26}$$

Since the W coefficients with one variable equal to $\frac{1}{2}$ are simple algebraic func-

tions[9, 11], the terms in brackets can be regarded as known. (26) is then a recursion formula, in that it relates a Racah coefficient with one variable equal to $c + \frac{1}{2}$ to coefficients with variables c and $c - \frac{1}{2}$. Hence, from a table of W coefficients with one variable equal to $\frac{1}{2}$, we can generate all desired tables. In practice, this procedure proved of great value.

Further specialization of (26) is, of course, advisable in numerical calculations. Reference (11) contains such formulae.

Other relations result from different choices of the variable $\bar{c}$. In particular, the choice $\bar{c} = 1$ yields useful relations.

In the angular correlation problem involving pure multipole radiations only the Racah coefficient with two repeated parameters, $W(a\nu cd; ad)$, occurs. This coefficient bears an interesting relation to the Legendre polynomials[13]. In the limit of large a, c, d we have:

$$[(2a + 1)(2d + 1)]^{\frac{1}{2}} W(a\nu cd; ad) \to (-1)^{\nu} P_{\nu}(\cos\vartheta)$$

$$\cos\vartheta \equiv \frac{c(c+1) - d(d+1) - a(a+1)}{2ad} \tag{27}$$

The recursion relation when particularized for these Racah functions becomes:

$$W(a\nu cd; ad)W(a\nu cd; ad) = \sum_{\lambda} [(2\lambda + 1)W(a\lambda a\nu; a\bar{\nu})W(d\lambda d\nu; d\bar{\nu})]W(a\lambda cd; ad) \tag{28}$$

The identity in (28) is thus seen to be the analogon to the familiar expansion for a product of Legendre functions. In the limit of large a, d we find, incidentally, that the bracketed terms are related to the vector addition coefficients, that is,

$$[(2\lambda + 1)(2a + 1)]^{\frac{1}{2}} W(a\lambda a\nu; a\bar{\nu}) \to (-1)^{\bar{\nu}} (\nu\bar{\nu}00 \mid \nu\bar{\nu}\lambda 0) + 0(1/a) \tag{29}$$

Particularizing (28) further, by letting $\bar{\nu} = 1$, yields the recursion relation:

$$W(a\nu + 1cd; ad) = \frac{2\nu + 1}{(\nu + 1)} \frac{2a(a+1) + 2d(d+1) - 2c(c+1) - \nu(\nu+1)}{[(2a + 2 + \nu)(2a - \nu)(2d + 2 + \nu)(2d - \nu)]^{\frac{1}{2}}} \cdot W(a\nu cd; ad)$$

$$- \frac{\nu}{\nu + 1} \left[\frac{(2a + \nu + 1)(2a + 1 - \nu)(2d + \nu + 1)(2d + 1 - \nu)}{(2a + 2 + \nu)(2a - \nu)(2d + 2 + \nu)(2d - \nu)} \right]^{\frac{1}{2}} \cdot W(a\nu - 1cd; ad) \tag{30}$$

This relation proves quite convenient for obtaining algebraic forms and numerical values for the $W(a\nu cd; ad)$.

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It has come to the author's attention that Professor E. P. Wigner in an unpublished manuscript (1941) defined and discussed functions equivalent to the Racah coefficients.

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NODAL LINES OF A VIBRATING PLATE

By R. J. Duffin

1. Introduction. An important result in the theory of vibrations states that nodal lines cannot occur in the gravest mode of vibration of a membrane. The question has been raised by A. Weinstein whether the same statement is true for the transverse vibration of a thin flat plate which is clamped on its boundary.

Let $w(x, y)$ be the displacement of the plate in a normal mode of vibration. Then in the region R of the plate

$$\Delta^2 w = k^2 w. \tag{1}$$

Here Δ is the Laplacian operator and k is the (angular) frequency of vibration. The clamping condition is expressed by

$$w = \partial w/\partial n = 0 \tag{2}$$

on the boundary of R. The question is whether w is of one sign for the smallest eigenvalue k^2. This question is related to other properties of the plate:

A. In a recent paper G. Szegö [1]† has shown that for plates of a given area the circular plate has the lowest frequency of vibration. He derives this result under the explicit assumption that nodal lines are not present.

B. If w is a mode of vibration which does not change sign then the following upper and lower bounds are valid for the eigenvalue.

$$\inf v^{-1}\Delta^2 v \leqq k^2 \leqq \sup v^{-1}\Delta^2 v. \tag{3}$$

Here v is any function which satisfies the boundary conditions (2) and does not change sign in R. Thus (1) gives $\iint_R v(\Delta^2 w - k^2 w)\,dx\,dy = 0$ hence by Green's theorem $\iint_R (v^{-1}\Delta^2 v - k^2)vw\,dx\,dy = 0$. But vw does not change sign so (3) follows. Inequalities (3) were known to Weinstein. The corresponding inequalities for other differential equations have been considered by several writers [2, 3].

C. Let u be any function which satisfies the boundary conditions (2) and which is normalized, that is $\iint_R u^2\,dx\,dy = 1$. Then the Rayleigh type upper bound for the lowest frequency k_0 is

$$k_0^2 \leqq \iint_R (\Delta u)^2\,dx\,dy. \tag{4}$$

If it were possible to find an upper bound from (4) which was less than a lower bound given by (3) then the existence of nodal lines would be proved.

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† Numbers in brackets indicate references cited at the end of the paper.

D. Let $G(x, y, x', y')$ be the Green's function for the biharmonic equation, $\Delta^2 G = 0$ with the boundary conditions (2). Then the differential equation (1) may be converted to the integral equation

$$w(x', y') = k^2 \iint_R G(x', y', x, y)\, w(x, y)\, dx\, dy. \tag{5}$$

Let $z(x, y)$ be any normalized function, then according to the well known theory of integral equations.

$$k_0^{-2} \geqq \iiiint_R G(x', y', x, y)\, z(x, y)\, z(x', y')\, dx\, dy\, dx'\, dy'. \tag{5b}$$

The equality is attained if $z = w$, an eigenfunction corresponding to k_0. If G does not change sign (5b) clearly implies that w does not change sign. Thus if for some region it is known that the gravest mode of vibration does change sign then it follows that the Green's function also changes sign. The converse of this last statement is probably not true. In this connection it is of interest to note that for several regions it has been shown recently that the Green's function does change sign [4, 5, 6]. On the other hand the Green's function for the circle is of one sign.

Section 2 of this paper considers the solution of (1) for the semi-infinite strip; $0 \leqq x < \infty$, $-1 \leqq y \leqq 1$. The boundary conditions (2) are imposed only on the sides of the strip. For uniformly bounded solutions it is shown that nodal lines occur if k is sufficiently small. This reveals a fundamental difference between the plate and the membrane. The equation for the membrane is $\Delta w = -k^2 w$ and $w = e^{-cx} \cos(\pi y/2)$ is a solution if $c^2 = \pi^2/4 - k^2$. Clearly then if $k^2 \leqq \pi^2/4$ this solution is of one sign in the strip.

Section 3 considers a plate in the shape of a long rectangle attached to a sufficiently large circle. The circle is attached to insure that k_0 is sufficiently small. Then the result of Section 2 makes it plausible that the gravest mode of vibration changes sign.

2. The Semi-Infinite Strip. The main result of this paper is the

THEOREM. *Suppose that the function $w(x, y)$, not identically zero, satisfies the equation*

$$\partial^4 w/\partial y^4 + 2\partial^4 w/\partial y^2 \partial x^2 + \partial^4 w/\partial x^4 = k^2 w \tag{6}$$

for $0 \leqq x < \infty$, $-1 < y < 1$. In this region, w together with its derivatives up to the orders appearing in (6) *are continuous and $0(e^{cx})$ uniformly. The boundary conditions are*

$$w \to 0,\ \partial w/\partial y \to 0 \text{ as } y \to \pm 1. \tag{7}$$

If $0 < k < 4.635$ then w must change sign. The constant here is the best possible.

Proof. For $p > c$ let

$$f(p, y) = \int_0^\infty e^{-px} w(x, y)\, dx. \tag{8}$$

Then by (7) $f \to 0$ and $df/dy \to 0$ as $y \to \pm 1$.
Multiply (6) by e^{-px} and integrate from 0 to ∞.
Integration by parts gives

$$d^4f/dy^4 + 2p^2d^2f/dy^2 + (p^4 - k^2)f = g \tag{9}$$

where

$$g(y) = p^3w + p^2\partial w/\partial x + p\partial^2 w/\partial x^2 + \partial^3 w/\partial x^3 + 2p\partial^2 w/\partial y^2 + 2\partial^3 w/\partial x \partial y^2$$

evaluated at $x = 0$.

In what follows $f' = f^{(1)} = df/dy$ etc.

LEMMA 1. *The solution of the differential equation with constant coefficients*

$$F^{(4)} + c_3F^{(3)} + c_2F^{(2)} + c_1F^{(1)} + c_0F = g$$

such that $F(0) = F^{(1)}(0) = F^{(2)}(0) = F^{(3)}(0) = 0$ *is*

$$F(y) = \int_0^y h(y - t)g(t)\, dt. \tag{10}$$

Here h *is the solution of the homogeneous equation such that*

$$h(0) = h^{(1)}(0) = h^{(2)}(0) = 0; \qquad h^{(3)}(0) = 1.$$

The proof is obvious on direct substitution.

If $g = 0$ equation (9) has solutions of the form $e^{\pm ry}$ where $m^4 + 2p^2m^2 + p^4 = k^2$. Suppose first that $p^2 > k$ then $m = \pm ir$ and $m = \pm is$ where $r = (p^2 - k)^{\frac{1}{2}}$ and $s = (p^2 + k)^{\frac{1}{2}}$. Then independent solutions are $\cos ry$, $\cos sy$, $\sin ry$, and $\sin sy$. The following function clearly satisfies Lemma 1.

$$h(y) = (1/2k)(r^{-1} \sin ry - s^{-1} \sin sy). \tag{11}$$

Moreover $F' = \int_0^y h'(y - t)g(t)\, dt$, where $h'(y) = (\cos ry - \cos sy)/2k$. Thus the solution of (9) is

$$f = F + H \tag{12}$$

where H is some solution of the homogeneous equation.

If $w(x, y)$ satisfies the conditions of the theorem so also does

$$w(x, y) + w(x, -y).$$

This operation would not introduce a change of sign so it is sufficient to consider w to be an even function of y. Thus f and g are even functions. Since h is an odd function it follows that F is an even function. Then from (12) it is seen that H is an even function so $H = a \cos ry + b \cos sy$.

The boundary conditions $f(1) = f'(1) = 0$ give

$$a \cos r + b \cos s = -F(1), \qquad ar \sin r + bs \sin s = F'(1).$$

Solving for a and b gives

$$Da = F(1)\, s \sin s + F'(1) \cos s, \qquad Db = -F(1)\, r \sin r - F'(1) \cos r, \text{ where}$$

$$D = r \sin r \cos s - s \sin s \cos r. \tag{12b}$$

Clearly $f(p, 0) = a + b$ so

$$f(p, 0) = [F(1)(s \sin s - r \sin r) + F'(1)(\cos s - \cos r)]/D. \tag{13}$$

The functions $\cos ry$, $r \sin ry$, $r^{-1} \sin ry$ are even entire functions of r hence they are even entire functions of p. Likewise $\cos sy$, $s \sin sy$, $s^{-1} \sin sy$ are even entire functions of p. By an elementary theorem it follows that F and F' are entire functions of p. It is seen from (13) that $f(p, 0)$ is a ratio of entire functions so $f(p, 0)$ is meromorphic.

LEMMA 2. *Let* $r^2 = p^2 - k$, $s^2 = p^2 + k$, $0 < k < 4.635$. *Then* D *does not vanish for any value of* p *on the real or imaginary axis.*

Proof. Suppose $r \sin r \cos s = s \sin s \cos r$. Dividing both sides by $\cos s \cos r$ gives

$$r \tan r = s \tan s. \tag{14}$$

If $\cos s = 0$ then $s \sin s \neq 0$ so $\cos r = 0$. Thus $D = 0$ is equivalent to (14) provided that the possibility of both sides of (14) being infinite is considered. Clearly $r \tan r$ is an even function of r so $r \tan r = T(r^2)$ where the function T is single valued. Likewise $s \tan s = T(s^2)$. Since p is real or imaginary, r^2 and s^2 are real. Clearly for $x \geqq 0$, $T(x) = (x)^{\frac{1}{2}} \tan (x)^{\frac{1}{2}}$; for $x \leqq 0$, $T(x) = -(-x)^{\frac{1}{2}} \tanh (-x)^{\frac{1}{2}}$.

The following properties of the real function $y = T(x)$ are evident by inspection:

(a) simple zeros at $x = 0, \pi^2, (2\pi)^2, \cdots$

(b) simple poles at $x = (\pi/2)^2, (3\pi/2)^2, (5\pi/2)^2, \cdots$

(c) $T(-\infty) = -\infty$.

(d) $T'(x) > 0$ except at a pole, as $2T'(r^2) = (1 + r^{-1} \sin r \cos r)/\cos^2 r$.

Thus the inverse function $x = T^{-1}(y)$ has an infinite number of branches, each branch being defined for all real y. Consider two points (x_1, y) and (x_2, y) on different branches, that is $T(x_1) = T(x_2)$.

Now the distance $X = x_2 - x_1$ is an extremum (for finite y) only if the two branches have equal slopes at these points, that is $T'(x_1) = T'(x_2)$. It is easy to show that T satisfies the differential equation $2T' = 1 + (T + T^2)/x$. Thus for an extremum

$$[T(x_1) + T^2(x_1)]/x_1 = [T(x_2) + T^2(x_2)]/x_2.$$

But $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Hence for finite T this demands that $T + T^2 = 0$ on each side, giving $T = 0$ or $T = -1$. The possibility of $T = \pm\infty$ must also be considered. If $T(x) = 0$ then $x = 0, \pi^2, (2\pi)^2, \cdots$ so $X \geqq \pi^2 - 0 = 9.86$. If $T(x) = \pm\infty$ then $x = -\infty, (\pi/2)^2, (3\pi/2)^2, \cdots$ so $X \geqq (3\pi/2)^2 - (\pi/2)^2 = 2\pi^2$. If $T(x) = -1$ tables of tanh and tan give $x = -1.439, 7.831, 37.4, \cdots$ so $X \geqq 7.831 + 1.439 = 9.270$.

Let $x_2 = p^2 + k$ and $x_1 = p^2 - k$ then $x_2 - x_1 = 2k < 9.270$. The contradiction proves Lemma 2.

LEMMA 3. *If w satisfies the equation $\Delta^2 w = k^2 w$ in a neighborhood of a point (x_0, y_0) and the derivatives up to the orders appearing are continuous then w may be expanded in a power series in $(x - x_0)$ and $(y - y_0)$.*

Proof. Let $\Delta w = kq$ then $\Delta q = kw$. Thus $\Delta(w + q) = k(w + q)$ and $\Delta(w - q) = -k(w - q)$. More generally suppose $\Delta u + c^2 u = 0$ where c is any complex constant. Thus $v(x, y, z) = u(x, y)e^{cz}$ is harmonic in three dimensions. Hence v may be expanded in a power series in $(x - x_0)$, $(y - y_0)$ and z. In particular for $z = 0$, u is a power series in $(x - x_0)$ and $(y - y_0)$. Thus $w + q$ and $w - q$ are power series. Adding shows that $2w$ is a power series.

Returning to the proof of the theorem suppose that $w \geqq 0$ in the strip. For $y = 0$, (8) becomes

$$f(p, 0) = \int_0^\infty e^{-px}\, w(x, 0)\, dx. \tag{15}$$

A proof of the following property of the Laplace transform is given by Doetsch [8].

LEMMA 4. *If $w \geqq 0$ and the integral in* (15) *converges if $p > p_1$ but not if $p < p_1$ then f has a singularity at $p = p_1$.*

By equation (13) and Lemma 2 it follows that f could not have a singularity at any real point p_1. Thus the integral in (15) converges for all p.

The numerator in (13) is clearly of exponential growth at most, as $p \to -\infty$. The denominator D may be written $\cos r \cos s\,(r \tan r - s \tan s)$. It is not difficult to see that D does not approach zero as $p \to -\infty$. Thus $f = 0\ (e^{-bp})$ for some positive constant b as $p \to -\infty$. This is consistent with (15) only if $w(x, 0)$ vanishes for all $x > b$. By Lemma 3 it follows that $w(x, 0) = 0$ for all $x \geqq 0$.

Since $w(x, y)$ is an even function of y it follows that $\partial w/\partial y = 0$ for $y = 0$ and $x \geqq 0$. Let $w_1(x, y) = w(x/2, y/2 + \frac{1}{2})$. Then by what has been shown above it is seen that w_1 also satisfies the conditions of the theorem ($k_1^2 = k^2/16$). Applying the same argument to w_1 would lead to the conclusion that w vanishes identically on the lines $y = 0$ and $y = \pm\frac{1}{2}$. Continuing this process shows that w vanishes along any line $y = j/2^n$ where j and n are integers. By continuity it follows that w vanishes in the strip. The contradiction shows that w must change sign.

Now let k and p be arbitrary constants. By direct substitution it is apparent that

$$w = e^{-px}[(\cos ry/\cos r) - (\cos sy/\cos s)] \tag{16}$$

is a solution of (6) such that $w = 0$ for $y = \pm 1$. If k and p are such that $r \tan r = s \tan s$ it is clear that $\partial w/\partial y = 0$ for $y = \pm 1$. In particular this condition has been shown to be satisfied above for $k = 4.635$, $p = 1.79$, $r = 1.2i$, and $s = 2.8$. If the two terms of (16) are sketched on a graph it is geometrically obvious that w is positive for $-1 < y < 1$. This shows that the constant 4.635 is the best possible and completes the proof of the theorem.

The restrictions imposed on the derivatives of w in the statement of the theorem can be weakened. Thus if w satisfies (6) and (7) then so does

$$w_1 = \int_0^1 w(x + t, y)\, dt.$$

This smoothing operation may be repeated several times. By this device it is possible to remove the restriction that the derivatives of w appearing in (6) are bounded in a finite part of the strip. It would be of interest to know whether or not it is possible to remove the restriction on the rate of growth at ∞.

3. The Rectangle Plus Circle. The normal modes of vibration of a circular plate clamped on its boundary may be expressed in terms of Bessel functions. The frequencies of the circularly symmetric modes of vibration are determined by the roots of $J_0(x)I_0'(x) - J_0'(x)I_0(x) = 0$. The smallest positive root is $x = 3.20$. The angular frequency is given by $k_0 = (3.20/a)^2$ where a is the radius of the circle. If $a > 1.49$ then $k_0 < 4.635$.

Consider now a composite region consisting of a rectangle of width 2 which overlaps a circle of radius $a > 1.49$. According to Rayleighs principle a region has at least as low a frequency of vibration as any of its subregions. Thus the lowest frequency of such a clamped plate is less than 4.635. As the length of the rectangle is increased indefinitely it is plausible that the lowest mode of vibration remains uniformly bounded and approaches one of those modes of the infinite plate just considered. It is to be expected therefore, that a clamped plate in the shape of a long rectangle attached to a circle whose diameter is $3/2$ the width of the plate will have nodal lines in the gravest mode of vibration.

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CORRECTIONS TO THE PAPER, "ON SUPERSONIC FLOW PAST A SLIGHTLY YAWING CONE II"

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On p. 204, in the second (unnumbered) equation in §3, $\partial w/\partial\theta$ should be $\partial w/\partial\phi$.

On p. 205, in equation 16, $\gamma\rho_n/\rho$ should be $\gamma\rho_n/\bar{p}$. At the end of §3, the reference to §8 should be to §7.

On p. 206, line 6, $\bar{p}_n$ should be ρ_n. In equation 17, $\bar{u}' + u \tan\theta$ should be $\bar{u}' + \bar{u} \tan\theta$.

In the last line of p. 207, ρ should be $\bar{p}$; and in footnote 10, the reference should be to §6, not §7.

On p. 208, in the first line of equation 29, $\bar{u})s_n'$ should read $\bar{u}\}s_n'$.

On p. 209, in the (unnumbered) equation after 31, $\bar{s}$ should be $\bar{p}$ (in both occurrences), and a large brace $\Big\}$ should be inserted after the term $\bar{v}\cot^2\theta$. In the last formula on this page, the last } should be $\Big\}$.

On p. 211, the first factor $(\bar{p}/\bar{p})$ on the right of equation 39 should be $(\bar{p}/\bar{p})$, and, on the last line, the upper limit of integration should be θ_s.

After the paper was written, an article by A. Ferri, *Supersonic flow around circular cones* (National Advisory Committee for Aeronautics, Technical Note 2236, 1950) has pointed out that the flow in the neighbourhood of the solid cone ($\theta_s \leqq \Theta \leqq \theta^*$ in the notation of my paper) presents further complications; hence further investigation would be needed to give the pressure distribution on the solid cone correct to the second order. See G. B. Young and C. P. Siska, *Supersonic flow around cones at large yaw*, J. Aeronaut. Sci. vol. 19 (1952) pp. 111–119 and references there given (p. 142).

ERRATUM: ON THE NUMERICAL EVALUATION OF FREDHOLM INTEGRAL EQUATIONS WITH THE AID OF THE LIOUVILLE-NEUMANN SERIES[1]

BY CARL WAGNER

On the right-hand side of Eq. (10) the factor λ is missing. The correct form of Eq. (10) is

$$\chi^{(n+1)}(x) = \frac{\lambda\left[\int k^{(n+1)}(x)\,dx\right]\left[\int K(x,z)\,dz\right]\Big/\int dx}{1-\lambda\left\{\int\left[\int K(x,z)\,dz\right]dx\right\}\Big/\int dx}$$

1. C. WAGNER, *Journ. Math. Physics* **30**, 232 (1952).

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